

Estimating the scaling function of multifractal measures and multifractal random walks using ratios

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Abstract

In this paper we prove central limit theorems for bias reduced estimators of the structure function of several multifractal processes, namely multiplicative cascades, multifractal random measures, multifractal random walk and multifractal fractional random walk as defined by Ludena (2008). Previous estimators of the structure functions considered in the literature were severely biased with a logarithmic rate of convergence, whereas the estimators considered here have a polynomial rate of convergence.

1 Introduction

A random process X with stationary increments will be called multifractal if its scaling behaviour is characterized by a strictly concave function ζ , such that for a certain range of real numbers q

$$\mathbb{E}[|X(t) - X(s)|^q] = c(q)|t - s|^{\zeta(q)}.$$

If the function ζ is linear, then the process is said to be monofractal, as is the case for instance for the fractional Brownian motion (FBM) B_H , $0 < H < 1$, which is defined as a continuous centered Gaussian process such that $B_H(0) = 0$ and

$$\text{var}(B_H(t) - B_H(s)) = |t - s|^{2H}.$$

Then, for all $q > -1$, $\mathbb{E}[B_H(t) - B_H(s)|^q] = c(q)|t - s|^{qH}$, with $c(q) = \mathbb{E}[|B_H(1)|^q]$

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Several truly multifractal processes with stationary increments have been defined. The earliest one is the multiplicative cascade introduced by [Mandelbrot \(1974\)](#) and rigorously studied by [Kahane and Peyrière \(1976\)](#). These processes were generalized by [Barral and Mandelbrot \(2002\)](#), [Muzy and Bacry \(2002\)](#) and [Bacry and Muzy \(2003\)](#). The latter authors introduced multifractal random measures (MRM) and multifractal random walks (MRW) as changed time Brownian motion. [Ludeña \(2008\)](#) and [Abry et al. \(2009\)](#) introduced multifractal (fractional) random walks which are conditionally fractional Gaussian processes.

For all these processes, multifractality results from a distributional scaling property which can be written as

$$\{X(\lambda t), 0 \leq t \leq T\} \stackrel{law}{=} \{U_\lambda X(t), 0 \leq t \leq T\},$$

for $0 < \lambda < 1$, U_λ is a positive random variable independent of the process X such that $\mathbb{E}[U_\lambda^q] = \lambda^{\zeta(q)}$ for $q < q_{\max}$ a certain parameter depending on the process under consideration. For the models we will formally introduce in the sequel, it is defined as

$$q_{\max} = \sup\{q : \zeta(q) \geq 1\}.$$

It is also important to note that the fixed time horizon T beyond which this scaling property need not be true is finite, except for monofractal processes such as the FBM.

Given a multifractal process observed discretely on $[0, T]$, it is of obvious interest to be able to identify the scaling function ζ .

Let t_1, \dots, t_N , with $t_i - t_{i-1} = \Delta = T/N$ be a regular partition of $[0, T]$ (typically on a dyadic scale). Typically, for $q < q_{\max}$, $\zeta(q)$ is estimated by calculating logarithms of the empirical structure function

$$S_N(X, q) := \sum_{j=0}^{N-1} |\Delta X_j|^q,$$

where $\Delta X_j = X(j\Delta) - X((j-1)\Delta)$. Estimators of ζ can then be defined by

$$\begin{aligned} \hat{\zeta}_N(q) &:= 1 + \frac{\log_2(S_N(X, q))}{\log_2(\Delta)}, \\ \tilde{\zeta}_N(q) &:= 1 + \log_2 \left(\frac{S_N(X, q)}{S_{2N}(X, q)} \right). \end{aligned}$$

These estimators have been thoroughly dealt with for multiplicative cascades in [Ossiander and Waymire \(2000\)](#). The authors show that $\hat{\zeta}_M(q)$ and $\tilde{\zeta}_M(q)$ are consistent estimators of $\zeta(q)$ for $q < q_0$, where $q_0 < q_{\max}$ is the largest value of q such that

$$\zeta(q) - q\zeta'(q) < q + 1. \tag{1}$$

For $q > q_0$, $\hat{\zeta}_M(q)$ is seen to converge almost surely to a linear function of q . Moreover, conditional CLTs (where the limiting distribution is a mixture of normal laws) are seen to hold for suitably normalized versions of both estimators if $2q < q_0$.

However, the convergence rates for both these estimators are very different. The rate of convergence of $\hat{\zeta}_N(q)$ is of order $\log_2(N)$ because of the existence of a bias term, whereas that of $\tilde{\zeta}_N(q)$ is a power of N which depends on ζ .

In order to enlarge the domain of consistency of the estimators and obtain unconditional CLTS, the so-called mixed asymptotic framework has been introduced by allowing the number L of basic observations intervals to increase with N . In the case of multiplicative cascades and MRM, the processes over different intervals are independent. The observations are $X((jL+k)\Delta)$, $0 \leq j \leq L-1$, $0 \leq k \leq N-1$ and the estimators are now modified as follows

$$\begin{aligned}\hat{\zeta}_{L,N}(X, q) &:= 1 + \frac{\log_2(S_{L,N}(X, q))}{\log_2(\Delta)} \\ \tilde{\zeta}_{L,N}(X, q) &:= 1 + \log_2 \left(\frac{S_{L,N}(X, q)}{S_{L,N}(X, q)} \right) ,\end{aligned}$$

with

$$S_{N,L}(X, q) := \sum_{j=0}^{L-1} \sum_{k=0}^{N-1} |\Delta X_{jL+k}|^q ,$$

The mixed asymptotic framework for multiplicative cascades has been recently developed in (Bacry et al. (2010)). The authors show that if $L = \lfloor 2^{\chi} \rfloor$, where $\lfloor x \rfloor$ stands for greatest integer $m \leq x$ with $\chi > 0$, then $\hat{\zeta}_{N,L}(X, q)$ is consistent for $q < q_\chi$ where q_χ is the largest value of q such that

$$\zeta(q) - q\zeta'(q) < q + \chi + 1 , \quad (2)$$

Note that as χ tends to infinity, q_χ might be greater than q_{\max} , so we will only consider values of χ such that $q_\chi < q_{\max}$.

However, once again, there exists a bias term $b_n := \mathbb{E}[M_1^q]/n$, which entails slow convergence of the estimator. In analogy to the non mixed asymptotic framework it is reasonable to consider ratio based estimators such as $\tilde{\zeta}_{N,L}(X, q)$ in order to improve convergence rates. It turns out, as follows quite straightforwardly from the results of Bacry et al. (2010), that $\tilde{\zeta}_{N,L}(X, q) \rightarrow \zeta(q)$, a.s. for a dyadic partition, but the authors failed to prove a CLT (for $2q < q_\chi$). Almost sure convergence for dyadic partitions, or in probability for general partitions, of $\hat{\zeta}_{N,L}(X, q)$ has also been recently considered by Duvernois (2009) for $\chi \geq 0$ and X a Brownian MRW or a MRM. However the author does not prove CLTs nor establish convergence rates in either case.

The main goal of this paper is to obtain CLTS for the estimator $\tilde{\zeta}$ in the mixed asymptotic setting, for multiplicative cascades, multifractal random measures (MRM) and multifractal random walks (MRW) for $H \geq 1/2$. Our main results in all these cases state unconditional CLTS with polynomial rates of convergence, contrary to $\hat{\zeta}(q)$ which can only achieve logarithmic rates of convergence, and to the case $L = 1$ where only conditional CLTs can be obtained.

We will consider multiplicative cascades in Section 2, MRM in Section 3, and MRW in Section 4. Section 5 contains the technical parts of the proofs. To the best of our knowledge our results are the first to deal with the MRW in the case $H > 1/2$.

Properties of log-Laplace transforms

We conclude this introduction by gathering certain convexity properties of Laplace transforms that we will need. In all the models we consider, the function ζ can be expressed as $\zeta(q) = q - \psi(q)$ for a function ψ which is the log-Laplace transform of some random variable Y , i.e.

$$\psi(q) = \log \mathbb{E}[e^{qX}] ,$$

with $\psi(0) = \psi(1) = 1$. The function ψ is thus strictly convex and we can express q_0 and q_{\max} in terms of ψ :

$$q_0 = \max\{q : q\psi'(q) - \psi(q) < 1\} , \quad q_{\max} = \max\{q : \psi(q) < q - 1\} .$$

The convexity of ψ and $\psi(1) = 0$ implies that $q_{\max} > 1$ if and only if $\psi'(1) < 1$, and $\psi'(q_{\max}) > 1$. This in turn implies that $1 < q_0 < q_{\max}$. Indeed, if $q \geq q_{\max}$, then $\psi'(q) > 1$; the strict convexity of ψ and $\psi(1) = 0$ implies that $\psi(q) < \psi'(q)(q - 1)$, hence

$$q\psi'(q) - \psi(q) > \psi'(q) > 1$$

thus $q > q_0$. Also, $\psi'(1) < 1$ implies that $q_0 \geq 1$, since $\psi'(1) - \psi(1) = \psi'(1) < 1$.

Let now q_χ be defined as the largest q such that $q\psi'(q) - \psi(q) < 1 + \chi$. Since ψ is convex, the function $q\psi'(q) - \psi(q)$ is increasing, thus $q_\chi > q_0$.

For $q > 1$, we will also be interested in the positive and increasing function $p \mapsto \psi(pq) - p\psi(q)$. If $pq < q_\chi$, then, by convexity,

$$\begin{aligned} 0 < \psi(pq) - p\psi(q) &= p\psi(pq) - \psi(q) - (p-1)\psi(pq) \leq (p-1)pq\psi'(pq) - (p-1)\psi(pq) \\ &= (p-1)\{pq\psi'(pq) - \psi(pq)\} < (p-1)(1 + \chi) . \end{aligned} \quad (3)$$

For $p = 2$, if $2q < q_\chi$, this yields

$$0 < \psi(2q) - 2\psi(q) < 1 + \chi . \quad (4)$$

2 Multiplicative cascades

In this section we give a precise formulation of consistency results for $\tilde{\zeta}(q)$, whenever $q < q_\chi$, and a CLT whenever $2q < q_\chi$, in the case of multiplicative cascades. The results are a straightforward application of previous results of [Bacry et al. \(2010\)](#) and

Ossiander and Waymire (2000). However, they provide the basic framework for dealing with both MRM and MRW so will be dealt with in some detail.

Before we state the main results we shall introduce the basic framework for mixed asymptotics following Bacry et al. (2010). For any given n -uplet r and $i < n$ set $r|i = (r_1, \dots, r_i)$ and if s is an i -uplet and v an $n - i$ -uplet set $r = s * v$ to be the resulting n -uplet obtained by concatenation.

For each $j \in \mathbb{Z}$ and fixed T , set $I^{(j)} := [jT, (j+1)T]$. Over each $I^{(j)}$ we will construct an independent *multiplicative cascade* as defined in Mandelbrot (1974). For this consider a collection $\{W_r^{(j)}, r \in \{0, 1\}^n, n \geq 1, j \in \mathbb{Z}\}$ of independent random variables with common law W such that $\mathbb{E}[W] = 1$ and $\mathbb{E}[W \log_2 W] < 1$. For each $n \geq 1$ and $j \in \mathbb{Z}$, consider the random measure defined by

$$\lambda_n^{(j)}(I) = T2^{-n} \sum_{r \in \{0,1\}^n \cap I} \prod_{i=1}^n W_{r|i}^{(j)},$$

for any I a Borel subset of $\mathcal{T}^{\mathbb{N}}$ over $I^{(j)}$, and each $r = (r_1, \dots, r_n) \in \{0, 1\}^n$ is identified to the real number $\sum_{i=1}^n r_i 2^{n-i}$. It can be seen (see Kahane and Peyrière (1976), Ossiander and Waymire (2000) for details on the construction and main results) that there exists a random measure $\lambda_\infty^{(j)}$, such that

$$\mathbb{P}(\lambda_n^{(j)} \Rightarrow \lambda_\infty^{(j)} \text{ as } n \rightarrow \infty) = 1,$$

where \Rightarrow stands for vague convergence. The limiting measure verifies $\mathbb{E}[\lambda_\infty^{(j)}([0, T])] = T$. By construction $\lambda_\infty^{(j)}$ are independent random measures, defined over the disjoint intervals $I^{(j)}$. Set $\lambda_\infty := \sum_{j \in \mathbb{Z}} \lambda_\infty^{(j)}$.

Set $\mathcal{F}_n = \sigma\{W_r^{(j)}, r \in \{0, 1\}^n, j \in \mathbb{Z}\}$ and let $\Delta_{k,n}^{(j)} := [(j + k2^{-n})T, (j + (k+1)2^{-n})T]$, $k = 0, \dots, 2^n - 1$ be the k -th dyadic interval at level n , of the interval $I^{(j)}$. Then,

$$\lambda_\infty(\Delta_{k,n}^{(j)}) = 2^{-n} Z_{j,k,n} \prod_{i=1}^n W_{r_n(k)|i}^{(j)}, \quad (5)$$

where for each n , $Z_{j,k,n}$, $0 \leq k < 2^n$, $j \in \mathbb{Z}$, are i.i.d. random variables with the same distribution as $\lambda_\infty([0, 1])$ and independent of \mathcal{F}_n , and $r_n(k)$ is the dyadic representation of k , i.e. $k = \sum_{i=1}^n r_{n,i}(k) 2^{n-i}$ for $k < 2^n$. Moreover, $Z_{j,2k,n+1}$ and $Z_{j,2k+1,n+1}$ are independent of $Z_{j,k',n}$ for $k' \neq k$. The identity (5) straightforwardly yields the scaling property:

$$\mathbb{E}[\lambda_\infty^q(\Delta_{k,n}^{(j)})] = 2^{-n\zeta(q)} \mathbb{E}[\lambda_\infty^q([0, 1])].$$

with

$$\zeta(q) = q - \log_2(\mathbb{E}[W^q]).$$

It is shown in Kahane and Peyrière (1976) that for $q > 1$, the condition $\zeta(q) > 1$ implies $\mathbb{E}[\lambda_\infty^q([0, T])] < \infty$.

Example 2.1. Consider the log-normal cascade, where $\log_2 W = \mu + \sigma Z$ and $Z \sim \mathbf{N}(0, 1)$. The condition $\mathbb{E}[W] = 1$ implies that $\mu = -\sigma^2/2$. Then it is easily obtained that

$$\psi(q) = q(q-1)\sigma^2/2, \quad q_{\max} = (2/\sigma^2) \vee 1, \quad q_0 = \sqrt{2}/\sigma, \quad q_\chi = \sqrt{(2/\sigma^2)(1+\chi)}.$$

Denote

$$S_{L,n}(q) = \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} \lambda_\infty^q(\Delta_{k,n}^{(j)})$$

and

$$\hat{\zeta}(q) := 1 - \frac{\log_2(S_{L,n}(q))}{n}, \quad \tilde{\zeta}(q) = 1 + \log_2 \left(\frac{S_{L,n}(q)}{S_{L,n+1}(q)} \right).$$

Consistency

For each $n \geq 1$, let $\{\xi, \xi_{j,k,n}, 0 \leq j \leq L-1, 0 \leq k \leq 2^n-1\}$ be a collection of i.i.d. random variables, independent of \mathcal{F}_n . Define

$$\tilde{S}_{n,q} = 2^{-nq} \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} \prod_{i=1}^n \left(W_{r_n(k)|i}^{(j)} \right)^q \xi_{j,k,n}. \quad (6)$$

In [Bacry et al. \(2010\)](#) the following general result is shown to hold.

Proposition 2.1. *Assume that $q < q_\chi$ and there exists $\epsilon > 0$ such that $\mathbb{E}[\xi^{1+\epsilon}] < \infty$. If ξ is non negative, then*

$$L^{-1}2^{-n}2^{n\zeta(q)}(\tilde{S}_{n,q} - \mathbb{E}[\tilde{S}_{n,q}]) \rightarrow 0, \quad \text{a.s.}$$

Note that by construction $\mathbb{E}[\tilde{S}_{n,q}] = L2^n2^{-n\zeta(q)}\mathbb{E}[\xi]$, so that the above result yields the convergence $L^{-1}2^{-n}2^{n\zeta(q)}\tilde{S}_{n,q} \rightarrow \mathbb{E}[\xi]$, a.s. under the stated conditions. As a consequence, by the definition of $S_{L,n}(q)$, Proposition 2.1 yields

$$L^{-1}2^{-n}2^{n\zeta(q)}S_{L,n}(q) \rightarrow \mathbb{E}[\lambda_\infty^q([0, 1])] \quad (7)$$

a.s. for $q < q_\chi$. Then, clearly,

$$\hat{\zeta}(q) - \zeta(q) - \chi + \frac{\log_2 \mathbb{E}[\lambda_\infty^q([0, 1])]}{n} \rightarrow 0 \quad \text{a.s.}$$

and this implies that $\tilde{\zeta}(q) \rightarrow \zeta(q)$ a.s., so that

$$\tilde{\zeta}(q) \rightarrow \zeta(q) \quad \text{a.s.}$$

On the other hand, if $q > q_\chi$, then [Bacry et al. \(2010\)](#) show $\hat{\zeta}(q) \rightarrow \zeta(q_\chi)q$, which is a linear function of q . In this case $\tilde{\zeta}(q)$ is also not consistent as the normalized structure function tends to zero ([Ossiander and Waymire \(2000\)](#)).

Central limit theorem

Based on Proposition 2.1, it is also possible to obtain a CLT for $\tilde{\zeta}(q)$. We remark that in the mixed asymptotic framework the limiting variance is deterministic. The proof of the CLT follows from a series of corollaries of the following general result for the mixed framework which is a direct generalization of Proposition 4.1 in Ossiander and Waymire (2000) and Proposition 2.1. We first state some general notation. Let $\{\xi, \xi_{j,k,n}, 0 \leq j \leq L-1, 0 \leq k \leq 2^{n-1}, n \geq 0\}$ be as above and let $\tilde{S}_{n,q}$ be as in (6). Define now

$$V_{n,q} = 2^{-2nq} \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} \prod_{i=1}^n (W_{r_n(k)|i}^{(j)})^{2q},$$

$$R_{n,q} := \tilde{S}_{n,q} / V_{n,q}^{1/2}.$$

The following proposition is seen to hold true as a direct generalization of Proposition 4.1 in Ossiander and Waymire (2000), whenever $2q < q_\chi$.

Proposition 2.2. *If $\mathbb{E}[\xi_{j,k,n}] = 0$, $\mathbb{E}[\xi_{j,k,n}^2] = \sigma^2$ and if*

$$\sup_n \sup_{j,k} \mathbb{E} [|\xi_{j,k,n}|^{2(1+\delta)}] < \infty,$$

for some $\delta > 0$, then for $2q < q_\chi$ under the assumption $M([0, 1]) > 0$, a.s.

$$\lim_{n \rightarrow \infty} \mathbb{E} [e^{izR_{n,q}} \mid \mathcal{F}_n] = e^{-\sigma^2 z^2 / 2}$$

and $R_{n,q}$ converges weakly to the centered Gaussian law with variance σ^2 .

The proof follows exactly as that of Proposition 4.1 in Ossiander and Waymire (2000), using Proposition 2.1. The latter also yields that $L^{-1}2^{-n}2^{n\zeta(2q)}V_{n,q}$ converges to 1 a.s. We now have

Proposition 2.3. *If $2q < q_\chi$, then*

$$L^{-1/2}2^{-n/2}2^{n\zeta(2q)/2} \{S_{L,n}(q) - 2^{\zeta(q)-1}S_{L,n+1}(q)\} \rightarrow_d N(0, V(q)),$$

with

$$V(q) = \text{var} \left(Z_0^q - 2^{\zeta(q)-1-q} \{Z_1^q W_1^q + Z_2^q W_2^q\} \right)$$

and Z_1, Z_2 are i.i.d. with the same distribution as $\lambda_\infty([0, 1])$ and independent of W_1, W_2 which are i.i.d. with the same distribution as W and $Z_0 = (Z_1 W_1 + Z_2 W_2)/2$ has the same distribution as $\lambda_\infty([0, 1])$.

Proof. The proof follows from Proposition 2.2, by noting that $S_{L,n}(q) - 2^{\zeta(q)-1}S_{L,n+1}(q)$ can be expressed as

$$S_{L,n}(q) - 2^{\zeta(q)-1}S_{L,n+1}(q) = 2^{-nq} \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} \prod_{i=1}^n (W_{r_n(k)*i}^{(j)})^q \xi_{j,k,n} .$$

with

$$\xi_{j,k,n} = Z_{j,k,n}^q - 2^{\zeta(q)-1-q} \left\{ Z_{j,2k,n+1}^q W_{r_n(k)*0}^q + Z_{j,2k,n+1}^q W_{r_n(k)*1}^q \right\}$$

since $r_n(k) * 0 = r_{n+1}(2k)$ and $r_n(k) * 1 = r_{n+1}(2k+1)$. Indeed, the r.v.s $\xi_{j,k,n}$, $j \in \mathbb{Z}$, $0 \leq k < 2^n$ are i.i.d. (for each fixed n) and it clearly holds that $\mathbb{E}[\xi_{j,k,n}] = 0$, $\mathbb{E}[\xi_{j,k,n}^2] = V(q)$ and $\mathbb{E}[|\xi_{j,k,n}|^{2+\delta}] < \infty$, whenever $2q < q_{max}$ for small enough $\delta > 0$. \square

Thus we obtain,

Theorem 2.4. *Assume $2q < q_\chi$. Then*

$$2^{n(1+\chi+2\psi(q)-\psi(2q))/2} \{\tilde{\zeta}(q) - \zeta(q)\} \rightarrow_d N(0, V(q)/\mathbb{E}[\lambda_\infty^q([0, 1]))).$$

Proof. By Proposition 2.1 and (7), $S_{L,n+1}(q)2^{\zeta(q)-1}/S_{L,n}(q) \rightarrow 1$ a.s. so

$$\begin{aligned} \tilde{\zeta}(q) - \zeta(q) &= \log_2 \left(\frac{S_{L,n}(q)}{2^{\zeta(q)-1}S_{L,n+1}(q)} \right) = -\log_2 \left(1 - \frac{S_{L,n}(q) - 2^{\zeta(q)-1}S_{L,n+1}(q)}{S_{L,n}(q)} \right) \\ &= \frac{S_{L,n}(q) - 2^{\zeta(q)-1}S_{L,n+1}(q)}{S_{L,n}(q)} + \{1 + o_P(1)\} . \end{aligned}$$

The proof is concluded by applying Proposition 2.3 and noting that $2^{n(1+\chi+\psi(q)-\psi(2q))/2} \sim L2^n 2^{-n\zeta(q)} / L^{1/2} 2^{n/2} 2^{-n\zeta(2q)/2}$. \square

3 Multifractal random measure

Once again we are interested in the mixed asymptotic framework defined by the parameter χ . The main ideas dealt with in this section are very similar in spirit to those in Duvernet (2009), we include the proofs for completeness' sake, since they are very similar to those which will be developed to study Multifractal Random Walks.

We recall the main definition and properties of Multifractal Random Measures, hereafter MRM, following Bacry and Muzy (2003).

Start by defining for $l > 0$, $w_l(u) = P(A_l(u))$ and set

$$M(I) = \lim_{l \rightarrow 0} \int_I e^{w_l(u)} du ,$$

where I is any Borel set in \mathbb{R} . Here P is a set process over $\mathcal{S}^+ = \{(s, t), t > 0\}$ such that $P(A \cup B) = P(A) + P(B)$ and $P(A)$ and $P(B)$ are independent if $A \cap B = \emptyset$, and

$$\mathbb{E}[e^{qP(A)}] = e^{\psi(q)\mu(A)} \quad (8)$$

with $\mu(A) = \int_A t^{-2} ds dt$ and

$$A_l(u) = \{(s, t), u - (t/2 \wedge T/2) < s < u + (t/2 \wedge T/2), t > l\}.$$

It will be useful to note that

$$\mu(A_l(t)) = T + \log(T/l).$$

The function ψ is the log-Laplace transform, assumed to exist for $q < q^*$, for some $q^* > 1$, of the infinitely divisible random measure P . It is convex and satisfy $\psi(0) = \psi(1) = 0$. By the Lévy Khinchine representation Theorem, it can be expressed as

$$\psi(q) = \frac{\sigma^2}{2} + mq + \int_{-\infty}^{\infty} \{e^{qx} - 1 - x\mathbf{1}_{\{|x| \leq 1\}}\} \nu(dx),$$

where ν is the Lévy measure of P and satisfies

$$\int_{-\infty}^{\infty} (x^2 \wedge 1) \nu(dx) < \infty.$$

The assumption that $\psi(q)$ is finite for $q < q^*$ entails the following condition. For all $q < q^*$,

$$\int_1^{\infty} e^{qx} \nu(dx) < \infty.$$

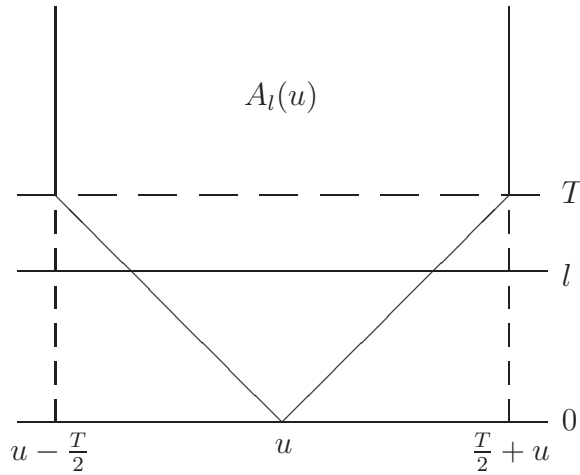


Figure 1: The set $A_l(u)$

By Theorem 4 in [Bacry and Muzy \(2003\)](#), there exists a certain infinitely divisible r.v. Ω_λ which is independent of $M[0, T]$, such that $\mathbb{E}[e^{q\Omega_\lambda}] = \lambda^{-\psi(q)}$ and for $\lambda, l \in (0, 1)$,

$$\{w_\lambda(\lambda u), 0 \leq u \leq T\} \stackrel{law}{=} \{w_l(u) + \Omega_\lambda, 0 \leq u \leq T\}. \quad (9)$$

The latter is known as the scaling property. This implies that

$$M([0, \lambda T]) \stackrel{d}{=} \lambda e^{\Omega_\lambda} M[0, T] \quad (10)$$

for $\lambda \in [0, 1]$, so that

$$\mathbb{E}[M^q([0, \lambda T])] = \lambda^{\zeta(q)} m(q) \quad (11)$$

with $\zeta(q) = q - \psi(q)$ and

$$m(q) = \mathbb{E}[M^q[0, T]].$$

It can be seen ([Bacry and Muzy \(2003\)](#)) that $\zeta(q) > 1 \Rightarrow \mathbb{E}[M^q([0, T])] < \infty$. As previously, set q_{\max} to be the greatest value of q such that $\zeta(q) \geq 1$ and for $\chi \geq 0$, define q_χ as the greatest value of q for which $q\psi'(q) < \psi(q) + 1 + \chi$. Assume moreover that χ is such that $q_\chi < q_{\max}$ and recall that by convexity, if $q < q_\chi$, then

$$\psi(q) - 2\psi(q/2) < 1 + \chi.$$

Example 3.1. Consider the Poisson cascade introduced by [Barral and Mandelbrot \(2002\)](#). Let N be a Poisson point process with intensity measure μ on $(-\infty, \infty) \times (0, \infty]$. Let Γ_i , $i \in \mathbb{Z}$ denote the points of N and let $\{W, W_i\}$ be a collection of i.i.d. positive random variables such that $\mathbb{E}[W] = 1$. Define the random measure P by

$$P(A) = \sum \log(W_i) \mathbf{1}_{\{\Gamma_i \in A\}}$$

for all relatively compact Borel sets $A \in (-\infty, \infty) \times (0, \infty]$. Then (8) holds with $\psi(q) = \mathbb{E}[W^q] - 1$.

$$q_{\max} = \max\{q : \mathbb{E}[W^q] \leq q\}, \quad q_\chi = \max\{q : q\mathbb{E}[W^q(\log(W) - 1)] \leq 1 + \chi\}.$$

Example 3.2. The random measure P can be a Gaussian random measure. Then $P(A) \sim \mathbf{N}(-\sigma^2\mu(A)/2, \sigma^2\mu(A))$ and $\psi(q) = \sigma^2 q(q-1)/2$ so that we get the same values of q_{\max} , q_0 and q_χ as for the multiplicative cascade of the previous section. Note that in this case, $\text{var}(P(A)) = \psi''(0)\mu(A)$ is finite if and only if $\mu(A) < \infty$.

Example 3.3. Let $\alpha \in (0, 1)$ and P be a totally skewed to the left α -stable random measure, i.e. $\psi(q) = \sigma^\alpha(q - q^\alpha)$. Then $q_{\max} > 1$ if and only if $\sigma^\alpha(1 - \alpha) < 1$ and then $q_{\max} = \infty$ and for $\chi \geq 0$, $q_\chi = \sigma^{-1}((1 + \chi)/(1 - \alpha))^{1/\alpha}$. It is noteworthy that contrary to the previous case, we have here that $\mathbb{E}[|P(A)|] = \infty$ for all A such that $\mu(A) > 0$, though $\mathbb{E}[|P(A)|^p] = c_{p,\alpha} \sigma^p \mu(A)^{p/\alpha}$ if $p < \alpha$ and $\mu(A) < \infty$.

Example 3.4. Let $\alpha \in (1, 2)$ and P be a totally skewed to the left α -stable random measure, i.e. $\psi(q) = \sigma^\alpha(q^\alpha - q)$. Then $q_{\max} > 1$ if and only if $\sigma^\alpha(\alpha - 1) < 1$ and then $q_{\max} < \infty$. For $\chi \geq 0$, $q_\chi = \sigma^{-1}((1 + \chi)/(\alpha - 1))^{1/\alpha}$.

With this notation, define as in the previous section $L = [2^{n\chi}]$, $\Delta_{k,n}^{(j)} = [(j + k2^{-n})T, (j + (k + 1)2^{-n})T]$ and

$$S_{L,n}(M, q) = \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} M^q(\Delta_{k,n}^{(j)}) ,$$

$$\tilde{\zeta}_M(q) = 1 + \log_2 \left(\frac{S_{L,n}(M, q)}{S_{L,n+1}(M, q)} \right) .$$

Consistency

For convenience, denote $\tau(q) = 1 + \zeta(q)$.

Proposition 3.1. *For $q < q_\chi$,*

$$L^{-1} 2^{n\tau(q)} S_{L,n}(M, q) \rightarrow m(q) , \quad a.s.$$

Plugging this into the definition of $\tilde{\zeta}_M(q)$ yields the consistency of $\tilde{\zeta}_M(q)$.

Corollary 3.2. *For $q < q_\chi$,*

$$\tilde{\zeta}_M(q) \rightarrow \zeta(q) , \quad a.s.$$

Central Limit Theorem

We next give a central limit Theorem for $\tilde{\zeta}_M(q)$ in the mixed asymptotic framework. Define the centered random variables

$$D_{j,k,n,q} := M^q(\Delta_{k,n}^{(j)}) - 2^{\tau(q)} (M^q(\Delta_{2k,n+1}^{(j)}) + M^q(\Delta_{2k+1,n+1}^{(j)})) \quad (12)$$

and $D_{j,n,q} = \sum_{k=0}^{2^n-1} D_{j,k,n,q}$. By construction, the variables $D_{j,k,n,q}$ are centered, and we can also bound their covariances. By stationarity and 2-dependence with respect to j , we can consider the case $j = 0$. The following bounds for the covariances is proved at the end of section 5.1.

Lemma 3.3. *If $2q < q_{\max}$, then for $k = 1, \dots, 2^{n-1}$,*

$$2^{n\zeta(2q)} \mathbb{E}[D_{0,0,n,q} D_{0,k,n,q}] = O(k^{-\{\psi(2q) - 2\psi(q) + 1\}}) . \quad (13)$$

We will start by proving a CLT for

$$D_{n,q} := \frac{\sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} D_{j,k,n,q}}{\sqrt{L} \sqrt{\mathbb{E}[D_{0,n,q}^2]}} .$$

Since the random variables $D_{j,n,q}$, $0 \leq j \leq L-1$ are 2 dependent, it suffices to show that for some $p > 1$,

$$\lim_{n \rightarrow \infty} \frac{L^{1-p} \mathbb{E}[D_{0,n,q}^{2p}]}{(\mathbb{E}[D_{0,n,q}^2])^p} = 0 . \quad (14)$$

Although the variables $D_{j,k,n,q}$ are conditionally $\mathcal{F}_{2^{-n}}$ i.i.d., unlike multiplicative cascades, they are not conditionally centered. Hence it is not possible to repeat the proof of the CLT for multiplicative cascades and it follows that it is necessary to require that $\chi > 0$.

We first need an expansion of $\mathbb{E}[D_{0,n,q}^2]$. Set

$$d_q = \mathbb{E} \left[\left(M^q([0, T] - 2^{\tau(q)} \{M^q([0, T/2]) + M^q([T/2, T])\}) \right)^2 \right] .$$

and $d_{k,q} = 2^{n\zeta(2q)} \mathbb{E}[D_{0,0,n,q} D_{0,k,n,q}]$. By the scaling property, $\mathbb{E}[D_{0,0,n,q}^2] = 2^{-n\zeta(2q)} d_q$ and $d_{k,q}$ does not depend on n . Then

$$\mathbb{E}[D_{0,n,q}^2] = 2^{-n\tau(2q)} d_q + 2 \cdot 2^{-n\tau(2q)} \sum_{k=1}^{2^n-1} (1 - k2^{-n}) d_{k,q} .$$

Since $\psi(2q) - 2\psi(q) > 0$, Lemma 3.3 implies that the series $\sum |d_{k,q}|$ is convergent, so the Cesaro mean above has a finite limit. Thus, there exists a constant Θ_q such that

$$\lim_{n \rightarrow \infty} 2^{n\tau(2q)} \mathbb{E}[D_{0,n,q}^2] = \Theta_q .$$

Next we prove (14) for $p = 2$, i.e. we compute the fourth moment of $D_{0,n,q}$, which exists if $4q < q_\chi$. By computations similar to those that yield Lemma 3.3, we can prove that

$$\mathbb{E}[D_{0,n,q}^{2p}] = O(2^{-n\tau(2pq)} + 2^{-np\tau(2q)}) .$$

The above discussion leads to the following result.

Proposition 3.4. *If $4q < q_\chi$, then*

$$D_{n,q} = \frac{\sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} D_{j,k,n,q}}{\sqrt{L \mathbb{E}[D_{0,n,q}^2]}} \rightarrow_d N(0, 1) ,$$

or equivalently,

$$L^{-1/2} 2^{n\tau(2q)/2} D_{n,q} \rightarrow_d N(0, \Theta_q) .$$

We can now prove the asymptotic normality of $\tilde{\tau}_M(q)$ and $\tilde{\zeta}_M(q)$. Denote

$$R_n = \frac{L^{-1}2^{n\tau(q)}\{S_{L,n}(M, q) - 2^{\tau(q)}S_{L,n+1}(M, q)\}}{L^{-1}2^{n\tau(q)}S_{L,n}(M, q)}.$$

By Proposition 3.1, $R_n = o(1)$, almost surely, so a second order Taylor expansion yields

$$\begin{aligned}\tilde{\zeta}_M(q) - \zeta(q) &= \log_2 \left(\frac{S_{L,n}(M, q)}{2^{\tau(q)}S_{L,n+1}(M, q)} \right) \\ &= \log \left(1 + \frac{L^{-1}2^{n\tau(q)}\{S_{L,n}(M, q) - 2^{\tau(q)}S_{L,n+1}(M, q)\}}{L^{-1}2^{n\tau(q)}S_{L,n}(M, q)} \right) \\ &= \frac{L^{-1}2^{n\tau(q)}\{S_{L,n}(M, q) - 2^{\tau(q)}S_{L,n+1}(M, q)\}}{L^{-1}2^{n\tau(q)}S_{L,n}(M, q)} + O_P(R_n^2).\end{aligned}$$

Note now that $S_{L,n}(M, q) - 2^{\tau(q)}S_{L,n+1}(M, q) = \sum_{j=1}^{L-1} \sum_{k=0}^{2^n-1} D_{j,k,n,q}$, so Proposition 3.4 yields the next result.

Theorem 3.5. *If $4q < q_\chi$ then*

$$2^{n(1+\chi-\psi(2q)+2\psi(q))/2}(\tilde{\zeta}_M(q) - \zeta(q)) \rightarrow N\left(0, \frac{\Theta_q}{m(q)}\right).$$

4 Multifractal random walk

Throughout this section, the MRM M and the process $\{w_l(u)\}$ will be as defined in the previous section. A multifractal random walk (MRW) is the process X obtained as the L^2 limit as $l \rightarrow 0$ of the integral $\int_0^t e^{w_l(u)} dB_H(u)$ where B_H is a standard fractional Brownian motion independent of M ; see Abry et al. (2009); Bacry et al. (2001); Bacry and Muzy (2003); Ludeña (2008). Recall that B_H is a continuous Gaussian, centered process with $B_H(0) = 0$ and

$$\text{var}(B_H(t) - B_H(s)) = |t - s|^{2H},$$

for all $t, s \in [0, 1]$. For $H = 1/2$, $B_{1/2}$ is the standard Brownian motion and will be simply denoted by B . This means that X is the conditionally (with respect to M) Gaussian process whose covariance function is defined in (15) or (16) below according to whether the Hurst parameter of the fBm $H = 1/2$ or $H > 1/2$. Except for the case $H = 1/2$, which is ordinary Brownian motion, it is worthwhile to remark that this conditionally Gaussian process X is not the time changed process $B_H(M[0, t])$.

Throughout this section \rightarrow_M will stand for conditional convergence in distribution given M and \mathbb{E}_M and var_M stand for the conditional expectation and variance given M . We consider the following two cases.

- Case $H = 1/2$ [Bacry et al. \(2001\)](#); [Bacry and Muzy \(2003\)](#). The MRW X is defined as the centered, conditionally Gaussian process with conditional covariance

$$\Gamma(s, t) = \lim_{l \rightarrow 0+} \int_0^{t \wedge s} e^{w_l(u)} du = M(s \wedge t). \quad (15)$$

The scaling function is $\zeta_{1/2}(q) = \zeta(q/2)$, since by (10) and (11), for $\lambda \in (0, 1)$,

$$\begin{aligned} \{X(\lambda t), 0 \leq t \leq T\} &\stackrel{law}{=} \lambda^{1/2} e^{\Omega_{\lambda/2}} \{X(t), 0 \leq t \leq T\}, \\ \mathbb{E}[|X(t)|^q] &= \mathbb{E}[\mathbb{E}_M[|X(t)|^q]] = c_q \mathbb{E}[M^{q/2}(t)] = c_q m(q/2) t^{\zeta(q/2)} \end{aligned}$$

where $c_q = \mathbb{E}[|\mathbf{N}(0, 1)|^q]$ and $m(q) = \mathbb{E}[M^q([0, 1])]$.

- Case $H > 1/2$ [Abry et al. \(2009\)](#); [Ludeña \(2008\)](#); [Muzy and Bacry \(2002\)](#). The MRW X is defined as the centered, conditionally Gaussian process with conditional covariance

$$\Gamma_H(s, t) = \lim_{l \rightarrow 0+} C_H \int_0^t \int_0^s \frac{e^{w_l(u)} e^{w_l(v)}}{|u - v|^{2-2H}} du dv = C_H \int_0^t \int_0^s \frac{M(du) M(dv)}{|u - v|^{2-2H}} \quad (16)$$

where $C_H = H(2H - 1)$. This process is well defined whenever $H - \psi(2)/2 > 1/2$, cf. [Ludeña \(2008\)](#). Convexity of ψ yields $\psi(2) > 0$. The scaling function ζ_H is defined by

$$\zeta_H(q) = qH - \psi(q),$$

since by (16) and (10) we have

$$\begin{aligned} \{X(\lambda t), 0 \leq t \leq T\} &\stackrel{law}{=} \lambda^H e^{\Omega_{\lambda}} \{X(t), 0 \leq t \leq T\}, \\ \mathbb{E}[|X(t)|^q] &= c_q m_H(q) t^{qH - \psi(q)} \end{aligned}$$

with

$$m_H(q) = \mathbb{E} \left[\left\{ \int_0^T \int_0^T |u - v|^{2H-2} M(du) M(dv) \right\}^{q/2} \right]. \quad (17)$$

Since we are considering the mixed asymptotic framework, we assume we have a collection of MRM $M^{(j)}$, $j = 0, \dots, L - 1$, which are independent, defined over consecutive intervals of length T . For $j = 0, \dots, L - 1$ and $k = 0, \dots, 2^n - 1$, define $\Delta X_{j,k,n} = X_{(j+(k+1)2^{-n})T} - X_{(j+k2^{-n})T}$. As above, we will investigate the asymptotic properties of $\tilde{\tau}_X(q)$ defined by

$$\tilde{\tau}_X(q) = \log_2 \left(\frac{S_{L,n}(X, q)}{S_{L,n+1}(X, q)} \right)$$

where now

$$S_{L,n}(X, q) = \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} |\Delta X_{j,k,n}|^q.$$

It will appear that $\tilde{\tau}_X(q)$ is an estimator of $\tau_H(q)$ defined for $H \geq 1/2$ by

$$\tau_H(q) = \zeta_H(q) - 1 .$$

Thus we define an estimator $\tilde{\zeta}_X(q)$ of the scaling function $\zeta_H(q)$ by

$$\tilde{\zeta}_X(q) = 1 + \tilde{\tau}_X(q) = 1 + \log_2 \left(\frac{S_{L,n}(X, q)}{S_{L,n+1}(X, q)} \right) .$$

Denote $T_n(X, q) = 2^{\tau_H(q)} S_{L,n+1}(X, q) - S_{L,n}(X, q)$. Then

$$\tilde{\tau}_X(q) - \tau_H(q) = \log \left(1 - \frac{T_n(X, q)}{S_{L,n}(X, q)} \right) .$$

We will prove that $T_n(X, q)/S_{L,n}(X, q) \rightarrow 0$, a.s. so that a Taylor expansion is valid and yields

$$\tilde{\tau}_X(q) - \tau_H(q) = -\frac{T_n(X, q)}{S_{L,n}(X, q)} (1 + o(1)) .$$

In order to study the ratio above, we will first prove that if $H = 1/2$, then

$$L^{-1} 2^{n\tau(q/2)} S_{L,n}(X, q) \rightarrow c_q m(q/2)$$

and if $H > 1/2$ then

$$L^{-1} 2^{n\tau_H(q)} S_{L,n}(X, q) \rightarrow c_q m_H(q)$$

with $m_H(q)$ as in (17) and $c_q = \mathbb{E}[|\mathbf{N}(0, 1)|^q]$ in both cases. To study $T_n(X, q)$, we write

$$T_n(X, q) = T_n(X, q) - \mathbb{E}_M[T_n(X, q)] + \mathbb{E}_M[T_n(X, q)] .$$

We will prove that in both cases, $T_n(X, q) - \mathbb{E}_M[T_n(X, q)]$ and $\mathbb{E}_M[T_n(X, q)]$ converge jointly to independent centered Gaussian distributions with the same normalization. This will yield the asymptotic normality of $\tilde{\zeta}_X(q) - \zeta_H(q)$.

Because of the different nature of the conditional dependence structure, which yields different scaling functions we will consider the cases $H = 1/2$ and $H > 1/2$ separately.

4.1 The case $H = 1/2$

In this case, it holds that

$$\begin{aligned} \mathbb{E}_M[S_{L,n}(X, q)] &= c_q S_{L,n}(M, q/2) , \\ \text{var}_M(S_{L,n}(X, q)) &= \sigma_q^2 S_{L,n}(M, q) . \end{aligned}$$

By Proposition (3.1), if $q < q_\chi$, we get

$$\begin{aligned} L^{-1}2^{n\tau(q/2)}\mathbb{E}_M[S_{L,n}(X, q)] &\rightarrow c_q m(q/2) , \quad \text{a. s.} \\ L^{-1}2^{n\tau(q)}\text{var}_M(S_{L,n}(X, q)) &\rightarrow \sigma_q^2 m(q) , \quad \text{a.s.} \end{aligned}$$

This implies that $L^{-1}2^{n\tau(q/2)}S_{L,n}(X, q)$ converges in probability to $c_q m(q/2)$. Since $S_{L,n}(X, q)$ is the sum of $L2^n$ conditionally independent terms, by Borel-Cantelli arguments similar to those used previously, almost sure convergence also holds, i.e.

$$L^{-1}2^{n\tau(q/2)}S_{L,n}(X, q) \rightarrow c_q m(q/2) , \quad \text{a.s.} \quad (18)$$

Using the notation (12) of the previous section, we have

$$\mathbb{E}_M[T_n(X, q)] = c_q 2^{\tau(q/2)} S_{L,n+1}(M, q/2) - c_q S_{L,n}(M, q/2) = -c_q \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} D_{j,k,n,q} .$$

Thus, by Proposition 3.4, if $q < q_\chi$ then $L^{-1/2}2^{n\tau(q)/2}\mathbb{E}_M[T_n(X, q)]$ converges to a centered Gaussian random variable with variance $\Sigma(1/2, q) = c_q^2 \Theta_{q/2}$. By the conditional independence of B and M , $T_n(X, q) - \mathbb{E}_M[T_n(X, q)]$ is a sum of centered and conditionally independent random variables with conditional variance

$$\text{var}_M(T_n(X, q)) = \sigma_q^2 S_{L,n}(M, q) + \sigma_q^2 (2^{2\tau(q/2)} - 2^{\tau(q/2)+1}) S_{L,n+1}(M, q) .$$

where $\sigma_q^2 = \text{var}(|\mathbf{N}(0, 1)|^q)$. By Proposition 3.1, $L^{-1}2^{n\tau(q)}\text{var}_M(T_n(X, q))$ converges almost surely to the positive constant $\Gamma(1/2, q)$ defined by

$$\Gamma(1/2, q) = \sigma_q^2 m(q) \{1 + (2^{2\tau(q/2)} - 2^{\tau(q/2)+1})2^{-\tau(q)}\} .$$

Thus, $L^{-1/2}2^{n\tau(q)/2}\{T_n(X, q) - \mathbb{E}_M[T_n(X, q)]\}$ converges weakly conditionally on M to a Gaussian random variable with variance $\Gamma(1/2, q)$, independent of M . Since the variance is deterministic, this assures non conditional convergence to the stated Gaussian r.v. Moreover, the conditional independence of B and M also implies that the sequence of random vectors

$$L^{-1/2}2^{n\tau(q)/2}(T_n(X, q) - \mathbb{E}_M[T_n(X, q)], \mathbb{E}_M[T_n(X, q)])$$

converges weakly to (Z_1, Z_2) where Z_1 and Z_2 are independent centered Gaussian random variables with respective variances $\Gamma(1/2, q)$ and $\Sigma(1/2, q)$.

The previous considerations yield

Theorem 4.1. *If $q < q_\chi$, then*

$$L^{1/2}2^{n(\psi(q/2)-\psi(q)/2+1/2)}\{\tilde{\zeta}_X(q) - \zeta_{1/2}(q)\} \rightarrow_d N\left(0, \frac{\Gamma(1/2, q) + \Sigma(1/2, q)}{c_q^2 m^2(q/2)}\right) ;$$

4.2 Case $H > 1/2$

We must first obtain an equivalent of (18). This is done in the following Lemma whose proof is postponed to Section 5.2.

Lemma 4.2.

- If $q < q_\chi$ then $L^{-1}2^{n\tau_H(q)}\mathbb{E}_M[S_{L,n}(X, q)] \rightarrow c_q m_H(q)$ a.s.
- If $2q < q_\chi$ and $H < 3/4$, there exists a positive constant $\Gamma(q)$ such that

$$L^{-2}2^{2n\tau_H(q)}\text{var}_M(S_{L,n}(X, q)) \rightarrow \Gamma(q), \quad \text{a. s.}$$

Moreover

$$L^{-1}2^{n\tau_H(q)}S_{L,n}(X, q) \rightarrow c_q m_H(q), \quad \text{a. s.} \quad (19)$$

Consider now the conditional variances

$$a_{j,k,n,H} = \mathbb{E}_M^{1/2}[(\Delta X_{j,k,n})^2], \quad \text{for } H \geq 1/2$$

and the conditionally standard Gaussian random variables

$$Y_{j,k,n} = \Delta X_{j,k,n} / a_{j,k,n,H}.$$

Let $G_q(x) = |x|^q - c_q$. With this notation

$$\begin{aligned} & T_n(X, q) - \mathbb{E}_M[T_n(X, q)] \\ &= \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} \left(2^{\tau_H(q)} \{ a_{j,2k,n+1,H}^q G_q(Y_{j,2k,n+1}) + a_{j,2k+1,n+1,H}^q G_q(Y_{j,2k+1,n+1}) \} - a_{j,n,l,H}^q G_q(Y_{j,k,n}) \right). \end{aligned}$$

Once again, arguing as in the proof of Lemma 4.2, one has the convergence

$$\text{var}_M(2^{n(2\psi(q)-\psi(2q)+1+\chi)/2} \{T_n(X, q) - \mathbb{E}_M[T_n(X, q)]\}) \rightarrow \Gamma(H, q), \quad \text{a. s.}$$

with $\Gamma(H, q)$ a certain positive constant. Following the proof of Theorem 3.1 in Ludeña (2008), the latter together with the bounds for the joint covariance structure of the conditional gaussian random variables $Y_{j,k,n}$ establish the following result. If $1/2 < H < 3/4$ and $2q < q_\chi$,

$$L^{-1/2}2^{-n\tau_H(2q)/2} \{T_n(X, q) - \mathbb{E}_M[T_n(X, q)]\} \rightarrow_M N(0, \Gamma(H, q)) \quad (20)$$

As for the case $H = 1/2$, the fact that $\Gamma(H, q)$ is deterministic establishes non conditional convergence in distribution.

Let us now study $\mathbb{E}_M[T_n(X, q)]$. We have

$$\mathbb{E}_M[T_n(X, q)] = c_q \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} (2^{\tau_H(q)} \{a_{j,2k,n+1,H}^q + a_{j,2k+1,n+1,H}^q\} - a_{j,k,n,H}^q) .$$

Denote $U_{n,k} = 2^{\tau_H(q)} \{a_{0,2k,n+1,H}^q + a_{0,2k+1,n+1,H}^q\} - a_{0,k,n,H}^q$. Then $U_{n,k}$ is centered and $\vartheta(q) = 2^{n\zeta_H(2q)} \text{var}(U_{n,0})$ does not depend on n . Similarly to Lemma 3.3, the following covariance bound holds.

Lemma 4.3.

$$2^{n\zeta_H(2q)} |\mathbb{E}[U_{0,n}, U_{k,n}]| \leq C k^{-\{\psi(2q)-2\psi(q)+1\}} . \quad (21)$$

We can now compute $\text{var}(\sum_{k=0}^{2^n-1} U_{n,k})$. By stationarity,

$$\text{var}\left(\sum_{k=0}^{2^n-1} U_{n,k}\right) = 2^{-n\tau_H(2q)} v_q + 22^{-n\tau_H(q)} \sum_{k=1}^{2^n-1} (2^n - k) 2^{n\zeta_H(2q)} \text{cov}(U_{n,0}, U_{n,k}) .$$

The series $\text{cov}(U_{n,0}, U_{n,k})$ is convergent, thus the Cesaro mean above converges to its sum, and we obtain that there exists a constant $\Sigma(H, q)$ such that

$$\begin{aligned} L^{-1} 2^{-n\tau_H(2q)} \text{var}(\mathbb{E}_M[T_n(X, q)]) &\rightarrow \Sigma(H, q) , \\ L^{-1/2} 2^{-n\tau_H(2q)/2} \mathbb{E}_M[T_n(X, q)] &\rightarrow_d \mathbf{N}(0, \Sigma(H, q)) . \end{aligned}$$

This and (20) yield the asymptotic normality of the estimator.

Theorem 4.4. *If $4q < q_\chi$ and $H < 3/4$ then*

$$2^{n(1+\chi-\psi(2q)+2\psi(q))/2} \{\tilde{\zeta}_X(q) - \zeta_H(q)\} \rightarrow_d \mathbf{N}\left(0, \frac{\Gamma(H, q) + \Sigma(H, q)}{c_q^2 m_H^2(q)}\right) .$$

5 Proofs

In all the proofs, without loss of generality, we set $T = 1$. We preface the proof by stating some results for infinitely divisible random measures. The infinitely divisible measure P introduced in Section 3 can be decomposed as $P = P_0 + P_1$ where P_0 and P_1 are independent and

$$\mathbb{E}[e^{qP_i(A)}] = e^{\mu(A)\psi_i(q)} ,$$

with

$$\begin{aligned} \psi_0(q) &= \frac{\sigma^2}{2} q^2 + mq + \int_{-1}^{\infty} \{e^{qx} - 1 - qx \mathbf{1}_{\{|x| \leq 1\}}\} \nu(dx) , \\ \psi_1(q) &= \int_{-\infty}^1 \{e^{qx} - 1\} \nu(dx) . \end{aligned}$$

Then, for A such that $\mu(A) < 1$ and $q \geq 1$, it holds that

$$\mathbb{E}[|e^{P_1(A)} - 1|^q] = O(\mu(A)) , \quad (22)$$

$$\mathbb{E}[|e^{P_0(A)} - 1 - P_0(A)|^q] = O(\mu(A)) . \quad (23)$$

Further, write

$$e^{P(A)} - 1 - P_0(A) = \{e^{P_1(A)} - 1\}e^{P_0(A)} + e^{P_0(A)} - 1 - P_0(A) . \quad (24)$$

This decomposition, (22), (23) and the independence of P_0 and P_1 yield

$$\mathbb{E}[|e^{P(A)} - 1 - P_0(A)|^q] = O(\mu(A)) . \quad (25)$$

Since P , P_0 and P_1 are independently scattered, these inequalities yield martingale maximal inequalities.

Lemma 5.1. *For A such that $\mu(A) \leq 1$, and for C_u an increasing sequence of measurable subsets of A , it holds that*

$$\mathbb{E}[\sup_u |P_0(C_u)|] = O(\mu^{1/2}(A)) , \quad (26)$$

$$\mathbb{E}[\sup_u |e^{P(C_u)} - 1|^q] = O(\mu(A)) , \quad q \geq 2 , \quad (27)$$

$$\mathbb{E}[\sup_u |e^{P(C_u)} - 1 - P_0(C_u)|^q] = O(\mu(A)) , q \geq 1 . \quad (28)$$

5.1 Multifractal random measure

Proof of Proposition 3.1. For $\varepsilon > 0$ such that $(1 + \varepsilon)q < q_\chi$ set $\varepsilon' = \min(2\varepsilon, 1/2, \chi)$, $n_0 = \lceil 1/\varepsilon' \rceil$, $\alpha = 1/n_0$ and finally $l_n = 2^{-(1-\alpha)n}$. Let

$$T_{n,q} = \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} e^{q w_{l_n}(j+2^{-n}k)} .$$

We will prove that there exist some constants $C, \eta > 0$, such that we have

$$\mathbb{E} \left[\left| \frac{T_{n,q}}{\mathbb{E}[T_{n,q}]} - 1 \right| \right] \leq C 2^{-n\eta} , \quad (29)$$

$$\mathbb{E} \left[\left| \frac{T_{n,q}}{\mathbb{E}[T_{n,q}]} - \frac{S_{L,n}(M, q)}{\mathbb{E}[S_{L,n}(M, q)]} \right| \right] \leq C 2^{-n\eta} . \quad (30)$$

The above inequalities and an application of the Borel-Cantelli lemma yield that

$$\frac{T_{n,q}}{\mathbb{E}[T_{n,q}]} \rightarrow 1, \text{ a.s.}$$

and

$$\frac{S_{L,n}(M, q)}{\mathbb{E}[S_{L,n}(M, q)]} - \frac{T_{n,q}}{\mathbb{E}[T_{n,q}]} \rightarrow 0, \text{ a.s.}$$

Since $\mathbb{E}[S_{L,n}(M, q)] = L 2^{-n\tau(q)} m(q)$, Proposition 3.1 follows. \square

Proof of (29). Set $0 < \chi' \leq \chi - \alpha$ for α as defined above and let $\delta > 0$ be such that $(1 + \delta)q < q_{\chi'}$. Now set $\epsilon = \inf(\delta, 1/4, \chi/2)$. Since $\chi' < \chi$, convexity of the function ψ assures $\epsilon \leq \delta < \epsilon$ and hence by construction we have that $2\epsilon < \alpha$.

We have, for all j, k, n ,

$$\mathbb{E}[M^q(\Delta_{k,n}^{(j)})] = 2^{-n\zeta(q)}m(q) ,$$

so that $\mathbb{E}[S_{L,n}(M, q)] = L2^{-n\tau(q)}m(q)$. Next, by (8),

$$\mathbb{E}[T_{n,q}] = L2^n \mathbb{E}[e^{qw_{l_n}(j+k2^{-n})}] = L2^n e^{\psi(q)} l_n^{-\psi(q)} .$$

The variables $e^{qw_{l_n}(j+2^{-n}k)} - \mathbb{E}[e^{qw_{l_n}(j+2^{-n}k)}]$ are 2-dependent (in j) and centered, so there exists a constant $C > 0$ such that

$$\mathbb{E} \left[\left| \frac{T_{n,q}}{\mathbb{E}[T_{n,q}]} - 1 \right|^{1+\epsilon} \right] \leq \frac{C}{L^\epsilon} \mathbb{E} \left[\left| \frac{1}{2^n} \sum_{k=0}^{2^n-1} \frac{e^{qw_{l_n}(2^{-n}k)}}{e^{\psi(q)} l_n^{-\psi(q)}} - 1 \right|^{1+\epsilon} \right] .$$

Applying Lemma 5.2, we have that for any $\epsilon' < \epsilon$, that there exists a constant C such that

$$\mathbb{E} \left[\left| \frac{T_{n,q}}{\mathbb{E}[T_{n,q}]} - 1 \right|^{1+\epsilon} \right] \leq C 2^{n[-\epsilon\chi + (1-\alpha)\{\psi((1+\epsilon)q) - (1+\epsilon)\psi(q) - \epsilon'\}]} .$$

By (3), for all $\epsilon > 0$ such that $q(1 + \epsilon) < q'_\chi$, we have

$$\begin{aligned} & -\epsilon\chi' + (1 - \alpha)(\psi((1 + \epsilon)q) - (1 + \epsilon)\psi(q) - \epsilon') \\ & < \epsilon[-\chi' + (1 - \alpha)(1 + \chi')] - (1 - \alpha)\epsilon' = -\alpha\epsilon(1 + \chi') + (1 - \alpha)(\epsilon - \epsilon') . \end{aligned}$$

This can be made negative since ϵ' can be chosen arbitrarily close to ϵ . This proves (29). \square

Proof of (30). We start by using again the argument of 2-dependence in j , to obtain, for ϵ and for some constant C ,

$$\begin{aligned} \mathbb{E} \left[\left| \frac{1}{L2^n} \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} \frac{M^q(\Delta_{k,n}^{(j)})}{2^{-n\zeta(q)}m(q)} - \frac{e^{qw_{l_n}(j+k2^{-n})}}{e^{\psi(q)} l_n^{-\psi(q)}} \right|^{1+\epsilon} \right] \\ \leq \frac{C}{L^\epsilon} \mathbb{E} \left[\left| \frac{1}{2^n} \sum_{k=0}^{2^n-1} \frac{M^q(\Delta_{k,n}^{(0)})}{2^{-n\zeta(q)}m(q)} - \frac{e^{qw_{l_n}(k2^{-n})}}{e^{\psi(q)} l_n^{-\psi(q)}} \right|^{1+\epsilon} \right] . \end{aligned}$$

For clarity, we now omit the superscript (0) in $\Delta_{k,n}^{(0)}$. Let M_n denote the random measure with density $e^{-w_{l_n}}$ with respect to M . By construction, the measure M_n is independent of

the process w_{l_n} and

$$\begin{aligned} M(\Delta_{k,n}) &= \int_{\Delta_{k,n}} e^{w_{l_n}(u)} M_n(du) = e^{w_{l_n}(k2^{-n})} \int_{\Delta_{k,n}} e^{w_{l_n}(u) - w_{l_n}(k2^{-n})} M_n(du) \\ &= M_n(\Delta_{k,n}) e^{w_{l_n}(k2^{-n})} + e^{w_{l_n}(k2^{-n})} \int_0^{2^{-n}} \{e^{w_{l_n}(u) - w_{l_n}(k2^{-n})} - 1\} M_n(du) . \end{aligned}$$

This yields

$$\begin{aligned} \frac{M^q(\Delta_{k,n})}{2^{-n\zeta(q)}m(q)} - \frac{e^{qw_{l_n}(k2^{-n})}}{e^{\psi(q)}l_n^{-\psi(q)}} &= \left\{ \frac{M_n^q(\Delta_{k,n})}{2^{-n\zeta(q)}m(q)e^{-\psi(q)}l_n^{\psi(q)}} - 1 \right\} \frac{e^{qw_{l_n}(k2^{-n})}}{e^{\psi(q)}l_n^{-\psi(q)}} \\ &\quad + \frac{M^q(\Delta_{k,n})}{2^{-n\zeta(q)}m(q)} - \frac{M_n^q(\Delta_{k,n})e^{qw_{l_n}(k2^{-n})}}{2^{-n\zeta(q)}m(q)} \\ &=: V_{n,k} + W_{n,k} . \end{aligned}$$

By (39) and (40) in Lemma 5.3 below, we obtain

$$\mathbb{E}[M_n^q(\Delta_{k,n})] = e^{-\psi(q)}2^{-n\zeta(q)}l_n^{\psi(q)}m_n(q)$$

with $\lim_{n \rightarrow \infty} m_n(q) = m(q)$ and

$$|m_n(q) - m(q)| \leq C2^{-\alpha n} . \quad (31)$$

Thus we can express $V_{n,k}$ as

$$\begin{aligned} V_{n,k} &= \left\{ \left(\frac{M_n^q(\Delta_{k,n})}{2^{-n\zeta(q)}m_n(q)e^{-\psi(q)}l_n^{\psi(q)}} - 1 \right) \frac{m_n(q)}{m(q)} + \frac{m_n(q)}{m(q)} - 1 \right\} \frac{e^{qw_{l_n}(k2^{-n})}}{e^{\psi(q)}l_n^{-\psi(q)}} \\ &=: \frac{m_n(q)}{m(q)} \mathbb{V}_{n,k} + \left\{ \frac{m_n(q)}{m(q)} - 1 \right\} \frac{e^{qw_{l_n}(k2^{-n})}}{e^{\psi(q)}l_n^{-\psi(q)}} . \end{aligned}$$

Denote $p = 1 + \epsilon$. By Minkowsky's inequality, we get

$$\begin{aligned} \mathbb{E}^{1/p} \left[\left| \frac{1}{2^n} \sum_{k=0}^{2^n-1} \frac{M^q(\Delta_{k,n}^{(0)})}{2^{-n\zeta(q)}m(q)} - \frac{e^{qw_{l_n}(k2^{-n})}}{e^{\psi(q)}l_n^{-\psi(q)}} \right|^{1+\epsilon} \right] \\ \leq \frac{m_n(q)}{m(q)} \mathbb{E}^{1/p} \left[\left| \frac{1}{2^n} \sum_{k=0}^{2^n-1} \mathbb{V}_{n,k} \right|^p \right] \end{aligned} \quad (32)$$

$$+ \left| \frac{m_n(q)}{m(q)} - 1 \right| \mathbb{E}^{1/p} \left[\left| \frac{1}{2^n} \sum_{k=0}^{2^n-1} \frac{e^{qw_{l_n}(k2^{-n})}}{e^{\psi(q)}l_n^{-\psi(q)}} \right|^p \right] \quad (33)$$

$$+ \mathbb{E}^{1/p} \left[\left| \frac{1}{2^n} \sum_{k=0}^{2^n-1} W_{n,k} \right|^p \right] . \quad (34)$$

The random variables $M_n(\Delta_{k,n})$ are $2^n l_n$ dependent and $e^{w_{l_n}}$ is independent of M_n , thus we have

$$\begin{aligned} \mathbb{E} \left[\left| \frac{1}{2^n} \sum_{k=0}^{2^n-1} \mathbb{V}_{n,k} \right|^p \right] &\leq \frac{C}{2^{(1-\alpha)\epsilon n}} \mathbb{E} \left[\left| \frac{M_n^q(\Delta_{0,n})}{2^{-n\zeta(q)} m_n(q) e^{-\psi(q)} l_n^{\psi(q)}} - 1 \right|^p \right] \mathbb{E} \left[\frac{e^{pq w_{l_n}(0)}}{e^{p\psi(q)} l_n^{-p\psi(q)}} \right] \\ &\leq C 2^{n\{\psi(pq) - p\psi(q) - (1-\alpha)\epsilon\}}. \end{aligned}$$

The bound (3) yields $\psi(pq) - p\psi(q) \leq \epsilon(1 + \chi')$, thus

$$\psi(pq) - p\psi(q) - (1 - \alpha)\epsilon - \epsilon\chi < \epsilon(\chi' - \chi + \alpha) \leq 0$$

because of the choice of $\chi' \leq \chi - \alpha$. We now bound the term in (33) by applying the bounds (31) and (38). We have

$$\left| \frac{m_n(q)}{m(q)} - 1 \right| \mathbb{E}^{1/p} \left[\left| \frac{1}{2^n} \sum_{k=0}^{2^n-1} \frac{e^{q w_{l_n}(k 2^{-n})}}{e^{\psi(q)} l_n^{-\psi(q)}} \right|^p \right] \leq C 2^{n\{(1-\alpha)\{\psi(pq) - p\psi(q) - \epsilon'\}/p - \eta\}}. \quad (35)$$

For the sum in (34), we use Jensen's inequality and the bound (41) of Lemma 5.3 to obtain

$$\mathbb{E} \left[\left| \frac{1}{2^n} \sum_{k=0}^{2^n-1} W_{n,k} \right|^p \right] \leq C \mathbb{E} \left[\left| \frac{M_n^q(\Delta_{0,n})}{2^{-n\zeta(q)}} - \frac{M_n^q(\Delta_{0,n}) e^{q w_{l_n}(0)}}{2^{-n\zeta(q)}} \right|^p \right] \leq 2^{-n\alpha/2} 2^{n\{\psi(pq) - p\psi(q)\}}. \quad (36)$$

In order to prove that the bounds (35) and (36) are good enough, we must now check that $\psi(pq) - p\psi(q) - \alpha s - \epsilon\chi < 0$ for small enough ϵ and $p = 1 + \epsilon$. Indeed we have proved earlier that $\psi(pq) - p\psi(q) < \epsilon(1 + \chi)$, so

$$\psi(pq) - p\psi(q) - \alpha/2 - \epsilon\chi < \epsilon - \alpha/2 < 0,$$

since $\epsilon < \alpha/2$. □

Lemma 5.2. *Let $\alpha = 1/n_0$ for some arbitrary integer $n_0 \geq 2$. For all $p > 1$ such that $\mathbb{E}[e^{pq w_{l_n}(0)}] < \infty$, for any $\epsilon' \in (0, p - 1)$, there exists a constant C such that*

$$\mathbb{E} \left[\left(\int_0^1 \frac{e^{q w_{l_n}(u)}}{\mathbb{E}[e^{q w_{l_n}(0)}]} du \right)^p \right] \leq C l_n^{-\{\psi(pq) - p\psi(q) - \epsilon'\}}. \quad (37)$$

$$\mathbb{E} \left[\left(2^{-n} \sum_{k=0}^{2^n-1} \frac{e^{q w_{l_n}(k 2^{-n})}}{\mathbb{E}[e^{q w_{l_n}(0)}]} \right)^p \right] \leq C l_n^{-\{\psi(pq) - p\psi(q) - \epsilon'\}}. \quad (38)$$

Proof. The choice of α implies that $(1 - \alpha)n_0 = n_0 - 1$ is an integer. Denote $g_n(u) = e^{q w_{l_n}(u)} / \mathbb{E}[e^{q w_{l_n}(0)}]$. Fix some integer k_0 , and define $n_1 = k_0 n_0$. If $n_1 < n$, then

$$\begin{aligned} \int_0^1 g_n(u) du &= \int_0^1 g_{n_1}(u) du + \int_0^1 \{g_n(u) - g_{n_1}(u)\} du \\ &= \int_0^1 g_{n_1}(u) du + \sum_{k=0}^{2^{(1-\alpha)n_1}-1} \int_{\Delta_{k, (1-\alpha)n_1}} \{g_n(u) - g_{n_1}(u)\} du \end{aligned}$$

We bound the first integral by applying Jensen's inequality:

$$\mathbb{E} \left[\left(\int_0^1 g_{n_1}(u) du \right)^p \right] \leq \mathbb{E}[g_{n_1}^p(0)] = 2^{(1-\alpha)n_1\{\psi(pq)-p\psi(q)\}} .$$

Since $w_{l_{n_1}}$ is independent of $w_{l_n} - w_{l_{n_1}}$, we can write

$$g_n(u) - g_{n_1}(u) = g_{n_1}(u) \left\{ \frac{e^{qw_{l_n}(u)-qw_{l_{n_1}}(u)}}{\mathbb{E}[e^{qw_{l_n}(0)-qw_{l_{n_1}}(0)}]} - 1 \right\}$$

Thus we see that the integrals $\int_{\Delta_{j,n_1}} \{g_n(u) - g_{n_1}(u)\} du$ are centered and 2-dependent conditionally on \mathcal{F}_{n_1} the sigma-field generated by $\{w_{l_{n_1}}(u), u \in [0, 1]\}$. Thus by [von Bahr and Esseen \(1965, Theorem 2\)](#), there is a constant C such that

$$\begin{aligned} \mathbb{E} \left[\left| \sum_{j=0}^{2^{(1-\alpha)n_1}-1} \int_{\Delta_{j,(1-\alpha)n_1}} \{g_n(u) - g_{n_1}(u)\} du \right|^p \right] &\leq C 2^{(1-\alpha)n_1} \mathbb{E} \left[\left| \int_{\Delta_{0,(1-\alpha)n_1}} \{g_n(u) - g_{n_1}(u)\} du \right|^p \right] \\ &\leq C 2^{p-1} 2^{(1-\alpha)n_1} \mathbb{E} \left[\left| \int_{\Delta_{0,(1-\alpha)n_1}} g_n(u) du \right|^p \right] + C 2^{p-1} 2^{(1-\alpha)n_1} \mathbb{E} \left[\left| \int_{\Delta_{0,(1-\alpha)n_1}} g_{n_1}(u) du \right|^p \right] \\ &\leq C 2^{p-1} 2^{(1-\alpha)n_1} \mathbb{E} \left[\left(\int_{\Delta_{0,(1-\alpha)n_1}} g_n(u) du \right)^p \right] + C 2^{p-1} 2^{\{1-p+\psi(pq)-p\psi(q)\}(1-\alpha)n_1} . \end{aligned}$$

Since $l_n/l_{n_1} = l_{n-n_1}$, By the scaling property (9), we have

$$\int_{\Delta_{0,(1-\alpha)n_1}} e^{qw_{l_n}(u)} du = l_{n_1} \int_0^1 e^{qw_{l_{n-n_1}l_{n_1}}(l_{n_1}u)} du \stackrel{law}{=} l_{n_1} e^{q\Omega_{l_{n_1}}} \int_0^1 e^{qw_{l_{n-n_1}}(u)} du .$$

Thus,

$$\begin{aligned} \mathbb{E} \left[\left(\int_{\Delta_{0,(1-\alpha)n_1}} g_n(u) du \right)^p \right] &= 2^{(1-\alpha)n_1(\psi(pq)-p)} \frac{(\mathbb{E}[e^{w_{l_{n-n_1}}(0)}])^p}{(\mathbb{E}[e^{w_{l_n}(0)}])^p} \mathbb{E} \left[\left(\int_0^1 g_{n-n_1}(u) du \right)^p \right] \\ &= 2^{(1-\alpha)n_1(\psi(pq)-p\psi(q)-p)} \mathbb{E} \left[\left(\int_0^1 g_{n-n_1}(u) du \right)^p \right] . \end{aligned}$$

Denote $u_n = \mathbb{E} \left[\left(\int_0^1 g_n(u) du \right)^p \right]$. We have proved the following recurrence:

$$u_n \leq B + C 2^{(1-\alpha)n_1(1-p+\psi(pq)-p\psi(q))} u_{n-n_1} .$$

By choosing k_0 large enough, this yields that for any $\epsilon' \in (0, \epsilon)$,

$$u_n \leq B + 2^{(1-\alpha)n_1(\psi(pq)-p\psi(q)-\epsilon')} u_{n-n_1} .$$

Thus, there exists a constant D such that

$$u_n \leq D 2^{(1-\alpha)n(\psi(pq)-p\psi(q)-\epsilon')} .$$

This proves (37). The bound (38) follows by replacing the measure du with a discrete measure. \square

Lemma 5.3. *Let $0 < \alpha < 1$. For $p \geq 1$, there exists a positive constant C such that*

$$\lim_{n \rightarrow \infty} 2^{n\zeta(q)} e^{\psi(q)} l_n^{-\psi(q)} \mathbb{E}[M_n^q(\Delta_{0,n})] = m(q) , \quad (39)$$

$$|\mathbb{E}[e^{qw_{l_n}(0)} M_n^q(\Delta_{0,n})] - \mathbb{E}[M^q(\Delta_{0,n})]| \leq C 2^{-\alpha n} 2^{-n\zeta(q)} , \quad (40)$$

$$\mathbb{E} \left[|e^{qw_{l_n}(0)} M_n^q(\Delta_{0,n}) - M^q(\Delta_{0,n})|^p \right] \leq C 2^{-\alpha n/2} 2^{-n\zeta(pq)} . \quad (41)$$

Proof. Define the sets $I_n, B_n(u), u \in [0, 2^{-n}]$ by

$$I_n = \bigcap_{0 \leq u \leq 2^{-n}} A_{l_n}(u) = A_{l_n(0)} \cap A_{l_n(2^{-n})} , \quad B_n(u) = A_{l_n}(u) \setminus I_n .$$

See Figure 2 for an illustration. By definition of the function ψ and the measure μ , we have, $\mathbb{E}[e^{qP(I_n)}] = e^{\psi(q)\mu(I_n)}$ and

$$\begin{aligned} \mu(I_n) &= \int_{I_n} \frac{ds dt}{t^2} = \int_{l_n}^1 \frac{t - 2^{-n}}{t^2} dt + \int_1^\infty \frac{1 - 2^{-n}}{t^2} dt \\ &= -\log(l_n) - 2^{-n}(l_n^{-1} - 1) + 1 - 2^{-n} = 1 - \log(l_n) - 2^{-\alpha n} = \mu(A_{l_n}(0)) - 2^{-\alpha n} . \end{aligned}$$

This yields

$$\mathbb{E}[e^{qP(I_n)}] = \mathbb{E}[e^{qw_{l_n}(0)}] \{1 + O(2^{-\alpha n})\} . \quad (42)$$

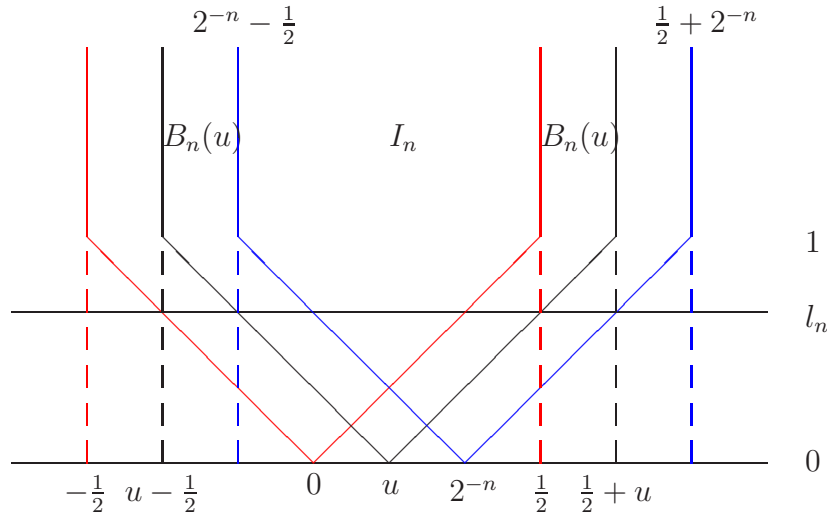


Figure 2: The sets I_n and $B_n(u)$

Then, $w_{l_n}(u) = P(I_n) + P(B_n(u))$, where the two summands are independent and we can write

$$\begin{aligned} M(\Delta_{0,n}) &= \int_0^{2^{-n}} e^{w_{l_n}(u)} M_n(du) = e^{P(I_n)} \int_0^{2^{-n}} e^{P(B_n(u))} M_n(du) \\ &= \xi_n \int_0^{2^{-n}} e^{P(B_n(u))} \bar{M}_n(du), \end{aligned} \quad (43)$$

with $\xi_n = e^{P(I_n)} M_n(\Delta_{0,n})$ and $\bar{M}_n(du) = M_n(du)/M_n(\Delta_{0,n})$ is a probability measure on $\Delta_{0,n}$. Note that $P(I_n)$, $M_n(\Delta_{0,n})$ and the integrand in the integral in (43) are mutually independent. This yields

$$\begin{aligned} M^q(\Delta_{0,n}) &= \xi_n^q \left(\int_0^{2^{-n}} e^{P(B_n(u))} \bar{M}_n(du) \right)^q \\ &= \xi_n^q + q \xi_n^q \int_0^{2^{-n}} P_0(B_n(u)) \bar{M}_n(du) \\ &\quad + q \xi_n^q \int_0^{2^{-n}} \{e^{P(B_n(u))} - 1 - P_0(B_n(u))\} \bar{M}_n(du) \\ &\quad + \xi_n^q \left\{ \left(\int_0^{2^{-n}} e^{P(B_n(u))} \bar{M}_n(du) \right)^q - 1 - q \int_0^{2^{-n}} \{e^{P(B_n(u))} - 1\} \bar{M}_n(du) \right\}. \end{aligned}$$

By elementary calculus, it is seen that for $q \geq 1$, there exists a constant C_q such that, for all $x > -1$,

$$0 \leq (1+x)^q - 1 - qx \leq C_q(x^2 + |x|^{q \vee 2}). \quad (44)$$

Applying this bound and (27) and the fact that $\mu(B_n(0)) = \mu(B_n(2^{-n})) = 2^{-\alpha n}$, we obtain, for $p \geq 1$,

$$\begin{aligned} \mathbb{E} \left[\xi_n^{pq} \left| \left(\int_0^{2^{-n}} e^{P(B_n(u))} \bar{M}_n(du) \right)^q - 1 - q \int_0^{2^{-n}} \{e^{P(B_n(u))} - 1\} \bar{M}_n(du) \right|^p \right] \\ \leq C_q \mathbb{E}[\xi_n^{pq}] \mathbb{E} \left[\sup_{u \in [0, 2^{-n}]} |e^{P(B_n(u))} - 1|^{2p} \right] \leq C 2^{-\alpha n} \mathbb{E}[\xi_n^{pq}]. \end{aligned}$$

Similarly, applying the bound (28), we obtain

$$\begin{aligned} \mathbb{E} \left[\xi_n^{pq} \left| \int_0^{2^{-n}} \{e^{P(B_n(u))} - 1 - P_0(B_n(u))\} \bar{M}_n(du) \right|^p \right] \\ \leq C_q \mathbb{E}[\xi_n^{pq}] \mathbb{E} \left[\sup_{u \in [0, 2^{-n}]} |e^{P(B_n(u))} - 1 - P_0(B_n(u))|^p \right] \leq C 2^{-\alpha n} \mathbb{E}[\xi_n^{pq}]. \end{aligned}$$

By the independence of $P_0(B_n(u))$ and the measure \bar{M}_n , we also have

$$\mathbb{E} \left[\xi_n^q \int_0^{2^{-n}} P_0(B_n(u)) \bar{M}_n(du) \right] = \mathbb{E} \left[\xi_n^q \int_0^{2^{-n}} \mathbb{E}[P_0(B_n(u))] \bar{M}_n(du) \right] \leq 2^{-\alpha n} \mathbb{E}[\xi_n^q] .$$

We have now proved that

$$\mathbb{E}[M^q(\Delta_{0,n})] = \mathbb{E}[\xi_n^q] \{1 + O(2^{-\alpha n})\} .$$

Moreover, applying (42) and the independence properties, we have

$$\mathbb{E}[\xi_n^q] = \mathbb{E}[e^{qP(I_n)}] \mathbb{E}[M_n^q(\Delta_{0,n})] = \mathbb{E}[e^{qw_{I_n}(0)}] \mathbb{E}[M_n^q(\Delta_{0,n})] \{1 + O(2^{-\alpha n})\} .$$

The last two bounds yield (40). By the scaling property, we have $\mathbb{E}[M^q(\Delta_{0,n})] = 2^{-n\zeta(q)} m(q)$. Since $\mathbb{E}[e^{qw_{I_n}(0)}] = e^{\psi(q)} l_n^{-\psi(q)}$, (39) follows.

To prove (41), we use the bound (26) to obtain

$$\mathbb{E} \left[\xi_n^{pq} \left| \int_0^{2^{-n}} P_0(B_n(u)) \bar{M}_n(du) \right|^p \right] \leq C 2^{-\alpha n/2} \mathbb{E}[\xi_n^{pq}] .$$

Gathering this and the previous bounds yields (41). \square

Proof of Lemma 3.3. By the scaling property, and since the random variables $D_{0,k,n,q}$ are centered, we can write

$$\begin{aligned} 2^{n\zeta(2q)} \mathbb{E}[D_{0,0,n,q} D_{0,k,n,q}] &= k^{\zeta(2q)} \text{cov} \left(M^q([0, \frac{1}{k}]), M^q([1 - \frac{1}{k}, 1]) \right) \\ &\quad - 2^{\tau(q)} (k - \frac{1}{2})^{\zeta(2q)} \text{cov} \left(M^q([0, \frac{1}{k-1/2}]), M^q([1 - \frac{1}{2k-1}, 1]) \right) \\ &\quad - 2^{\tau(q)} k^{\zeta(2q)} \text{cov} \left(M^q([0, \frac{1}{k}]), M^q([1 - \frac{1}{2k}, 1]) \right) \\ &\quad - 2^{\tau(q)} k^{\zeta(2q)} \text{cov} \left(M^q([0, \frac{1}{2k}]), M^q([1 - \frac{1}{k}, 1]) \right) \\ &\quad + 2^{2\tau(q)} (k - 1/2)^{\zeta(2q)} \text{cov} \left(M^q([0, \frac{1}{2k-1}]), M^q([1 - \frac{1}{2k-1}, 1]) \right) \\ &\quad + 2^{2\tau(q)} k^{\zeta(2q)} \text{cov} \left(M^q([0, \frac{1}{2k}]), M^q([1 - \frac{1}{2k}, 1]) \right) \\ &\quad - 2^{\tau(q)} k^{\zeta(2q)} \text{cov} \left(M^q([\frac{1}{2k}, \frac{1}{k}]), M^q([1 - \frac{1}{k}, 1]) \right) \\ &\quad + 2^{2\tau(q)} (k - 1/2)^{\zeta(2q)} \text{cov} \left(M^q([\frac{1}{2k-1}, \frac{2}{2k-1}]), M^q([1 - \frac{1}{2k-1}, 1]) \right) \\ &\quad + 2^{2\tau(q)} k^{\zeta(2q)} \text{cov} \left(M^q([\frac{1}{2k}, \frac{1}{k}]), M^q([1 - \frac{1}{2k}, 1]) \right) . \end{aligned}$$

We will prove below that each covariance terms that appear above is of order $k^{-2\zeta(q)-1}$, which yields $2^{n\zeta(2q)}\mathbb{E}[D_{0,0,n,q}D_{0,k,n,q}] = O(k^{\zeta(2q)-2\zeta(q)-1})$, and since $\zeta(2q) - 2\zeta(q) = 2\psi(q) - \psi(2q)$, the bound (13) is proved.

Let us now prove the bounds for each covariances. The derivation is similar for all of them, so we will only study one explicitly, say $\text{cov}([M^q([0, 1/k]), M^q([1 - 1/k, 1])]$.

For $k \geq 3$, define $l = 1 - 2/k$ and $M_l(du) = e^{-w_l(u)}M(du)$. By construction, the measure M_l is independent of $\{w_l(u)\}$ and $M_l([0, 1/k])$ is independent of $M([1 - 1/k, 1])$. Define the sets A_k and B_k by

$$A_k = A_l(1/k) \setminus A_l(1 - 1/k), \quad B_k = A_l(1 - 1/k) \setminus A_l(1/k),$$

For $u \leq k$ and $v \geq 1 - 1/k$, define

$$\begin{aligned} C_k^{u,v} &= A_l(u) \cap A_l(v), \\ D_{k,u} &= C_k^{1/k,v} \setminus C_k^{u,v}, \quad D'_{k,u} = A_l(u) \setminus A_l(1/k) \\ E_{k,u} &= C_k^{u,1-1/k} \setminus C_k^{u,v}, \quad E'_{k,u} = A_l(v) \setminus A_l(1 - 1/k). \end{aligned}$$

See Figure 3 for an illustration.

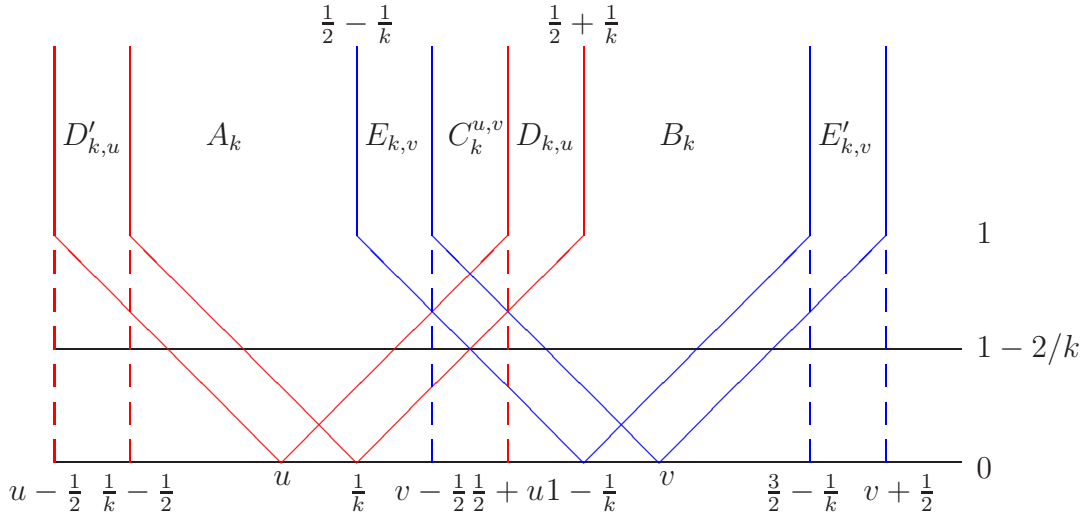


Figure 3: The sets A, B, C, D, D', E, E'

Note that all these sets are above the horizontal line at level $l = 1 - 1/2k$, hence $P(A)$ is independent of M_l and $P(A)$ is independent of $P(B)$, where A, B are any two of these sets. Note that

$$\cup_{u \leq 1/k, v \geq 1-1/k} C_k^{u,v} = C_k^{1/k, 1-1/k},$$

and let this set be simply denoted C_k . Note that $D_{k,u} \subset C_k$, $E_{k,v} \subset C_k$, $D'_{k,u} \subset D'_{k,0}$ and $E'_{k,v} \subset E'_{k,1}$. We moreover have

$$\begin{aligned}\mu(A_k) &= \mu(B_k) = 1, \\ \mu(C_k) &= -\log(1 - 2/k), \\ \mu(D'_{k,0}) &= \mu(E'_{k,1}) = 1/(k-2).\end{aligned}$$

With this notation, we have, for $u \leq 1/k$ and $v \geq 1 - 1/k$,

$$\begin{aligned}w_l(u) &= P(A_k) + P(C_k^{u,v}) + P(D'_{k,u}) + P(E_{k,v}), \\ w_l(v) &= P(B_k) + P(C_k^{u,v}) + P(D_{k,u}) + P(E'_{k,v}),\end{aligned}$$

Recall that the random measure P can be split into two independent random measures P_0 and P_1 such that $P = P_0 + P_1$. For $i = 0, 1$ and $u \in [0, 1/k]$, define $\pi_{i,k}(u) = P_i(D'_{k,u}) + P_i(C_k^{u,1-1/k})$ and

$$\pi_k(u) = \pi_{0,k}(u) + \pi_{1,k}(u).$$

Similarly, for $i = 0, 1$ and $v \in [1 - 1/k, 1]$, define $\pi'_{i,k}(v) = P_i(E'_{k,v}) + P_i(C_k^{1/k,v})$ and

$$\pi'_k(v) = \pi'_{0,k}(v) + \pi'_{1,k}(v).$$

Let \bar{M}_l and \bar{M}'_l denote the normalized measures $M_l/M_l([0, 1/k])$ and $M_l/M_l([1 - 1/k, 1])$. Denote finally

$$\begin{aligned}\zeta_k &= M_l([0, 1/k]), \quad \xi_k = M_l([1 - 1/k, 1]), \\ \alpha_k &= \int_0^{1/k} \pi_{0,k}(u) \bar{M}_l(du), \quad \beta_k = \int_{1-1/k}^1 \pi'_{0,k}(v) \bar{M}'_l(dv), \\ \gamma_k &= \int_0^{1/k} \int_{1-1/k}^1 \{e^{\pi_k(u) + \pi'_k(v)} - 1 - \pi_{0,k}(u) - \pi'_{0,k}(v)\} \bar{M}_l(du) \bar{M}'_l(dv) \\ a_k &= \mathbb{E}[\zeta_k^q], \quad b_k = \mathbb{E}[\xi_k^q].\end{aligned}$$

With this notation, we obtain

$$\begin{aligned}\mathbb{E}[M^q([0, 1/k])M^q([1 - 1/k, 1])] &= \mathbb{E}[e^{qP(A_k)}]\mathbb{E}[e^{qP(B_k)}] \mathbb{E}\left[\zeta_k^q \xi_k^q \left(\int_0^{1/k} \int_{1-1/k}^1 e^{\pi_k(u) + \pi'_k(v)} \bar{M}_l(du) \bar{M}_l(dv)\right)^q\right] \\ &= e^{2\psi(q)} a_k b_k + q e^{2\psi(q)} \{\mathbb{E}[\zeta_k^q \alpha_k] + \mathbb{E}[\xi_k^q \beta_k] + \mathbb{E}[\zeta_k^q \xi_k^q \gamma_k]\} \\ &\quad + e^{2\psi(q)} \mathbb{E}\left[\zeta_k^q \xi_k^q \left\{ \left(\int_0^{1/k} \int_{1-1/k}^1 e^{\pi_k(u) + \pi'_k(v)} \bar{M}_l(du) \bar{M}_l(dv)\right)^q \right. \right. \\ &\quad \left. \left. - 1 - q \int_0^{1/k} \int_{1-1/k}^1 \{e^{\pi_k(u) + \pi'_k(v)} - 1\} \bar{M}_l(du) \bar{M}_l(dv) \right\} \right]\end{aligned}$$

Since π_0 and π'_0 are independent of the measure M_l , we have

$$\mathbb{E}[\zeta_k^q \alpha_k] = \mathbb{E} \left[\zeta_k^q \int_0^{1/k} \mathbb{E}[\pi_{0,k}(u)] \bar{M}_l(du) \right] \leq a_k \{\mu(D'_{k,0}) + \mu(C_k)\} = O(a_k/k) .$$

Similarly, $\mathbb{E}[\xi_k^q \beta_k] = O(b_k/k)$. Applying the bound (28) (and some algebra), we obtain

$$|\mathbb{E}[\zeta_k^q \xi_k^q \gamma_k]| \leq a_k b_k \mathbb{E} \left[\sup_{u \in [0, 1/k]} \sup_{v \in [1-1/k, 1]} |e^{\pi_k(u) + \pi'_k(v)} - 1 - \pi_{0,k}(u) - \pi'_{0,k}(v)| \right] = O(a_k b_k/k) .$$

For the last term, we apply the bounds (44) and (27) and obtain

$$\begin{aligned} \mathbb{E} \left[\zeta_k^q \xi_k^q \left\{ \left(\int_0^{1/k} \int_{1-1/k}^1 e^{\pi_k(u) + \pi'_k(v)} \bar{M}_l(du) \bar{M}_l(dv) \right)^q \right. \right. \\ \left. \left. - 1 - q \int_0^{1/k} \int_{1-1/k}^1 \{e^{\pi_k(u) + \pi'_k(v)} - 1\} \bar{M}_l(du) \bar{M}_l(dv) \right\} \right] = O(a_k b_k/k) . \end{aligned}$$

Altogether, we obtain

$$\mathbb{E}[M^q([0, 1/k]) M^q([1 - 1/k, 1])] = e^{2\psi(q)} a_k b_k \{1 + O(1/k)\} . \quad (45)$$

We also have

$$\begin{aligned} M([0, 1/k]) &= e^{P(A_k)} \zeta_k \times \left\{ 1 + \alpha_k + \int_0^{1/k} \{e^{\pi_k(u)} - 1 - \pi_{0,k}(u)\} \bar{M}_l(du) \right\} , \\ M([1/k, 1]) &= e^{P(B_k)} \xi_k \times \left\{ 1 + \beta_k + \int_{1-1/k}^1 \{e^{\pi'_k(v)} - 1 - \pi'_{0,k}(v)\} \bar{M}'_l(dv) \right\} . \end{aligned}$$

By similar techniques, we obtain

$$\begin{aligned} \mathbb{E}[M^q([0, 1/k])] &= e^{\psi(q)} a_k \{1 + O(1/k)\} , \\ \mathbb{E}[M^q([1 - 1/k, 1])] &= e^{\psi(q)} b_k \{1 + O(1/k)\} , \end{aligned}$$

and thus

$$\mathbb{E}[M^q([0, 1/k])] \mathbb{E}[M^q([1 - 1/k, 1])] = e^{2\psi(q)} a_k b_k \{1 + O(1/k)\} . \quad (46)$$

Gathering (45) and (46) yields

$$\text{cov}(M_l^q([0, 1/K]), M_l^q([1 - 1/k, 1])) = O(a_k b_k k^{-1}) .$$

Finally, the previous bounds also imply that $a_k = O(k^{-\zeta(q)})$ and $b_k = O(k^{-\zeta(q)})$ so that we have finally proved that $\text{cov}(M^q[0, 1/k], M^q([1 - 1/k, 1])) = O(k^{-2\zeta(q)-1})$. \square

5.2 Multifractal random walk

As a first step we require the following result for $a_{j,k,n,H}$ which is analogous to Lemma 5.3. Set $\tilde{a}_{j,k,n,H} = e^{w_{l_n}(t_{j,k})} \tilde{\delta}_{j,k,n,H}$ with

$$\tilde{\delta}_{j,k,n,H}^2 = \int_{\Delta_{k,n}^{(j)}} \int_{\Delta_{k,n}^{(j)}} |u - v|^{2H-2} M_n(du) M_n(dv)$$

and for $j_1 \neq j_2$,

$$\tilde{\rho}_H(j_1, j_2, k, k') = \frac{\int_{\Delta_{k,n}^{(j_1)}} \int_{\Delta_{k',n}^{(j_2)}} |u - v|^{2H-2} M_n(du) M_n(dv)}{\delta_{j_1,k,n,H} \delta_{j_2,k',n,H}}.$$

Lemma 5.4. *For $p \geq 1$ such that $2pq < q_\chi$ and for $r \geq 2$, there exist $\eta, C > 0$ and uniformly bounded constants $c_{q,H}(k, k')$ such that*

$$\left| 2^{n\zeta_H(2q)} e^{\psi(2q)} l_n^{-\psi(2q)} \mathbb{E}[\tilde{\delta}_{j,k,n,H}^{2q}] - m_H(2q) \right| = O(2^{-n\eta}), \quad (47)$$

$$\left| 2^{n\zeta_H(2q)} e^{\psi(2q)} l_n^{-\psi(2q)} \mathbb{E}[\tilde{a}_{0,k,n,H}^q \tilde{a}_{0,k',n,H}^q] - c_{q,H}(k, k') \right| = O(2^{-n\eta}), \quad (48)$$

$$2^{n\zeta_H(2pq)} \mathbb{E}[|a_{0,k,n,H}^{2q} - \tilde{a}_{0,k,n,H}^{2q}|^p] = O(2^{-n\eta}), \quad (49)$$

$$2^{n\zeta_H(2pq)} \mathbb{E}[|a_{0,k,n,H}^q a_{j,k',n,H}^q - \tilde{a}_{0,k,n,H}^q \tilde{a}_{0,k',n,H}^q|^p] = O(2^{-n\eta}), \quad (50)$$

The proof is along the lines of the proof of Lemma 5.3 and is omitted. We can now prove Lemma 4.2.

Proof of Lemma 4.2. The first part follows by Lemma 5.4, along the same lines as the proof of Proposition 3.1. For the second part of the Lemma, where we assume $H < 3/4$, using the notation introduced in Section 4.2, we have

$$S_{L,n}(X, q) - \mathbb{E}_M[S_{L,n}(X, q)] = \sum_{j=0}^{L-1} \sum_{k=0}^{2^n} a_{j,k,n,H}^q G_q(Y_{j,k,n}).$$

Let $g_r(q)$ $r \geq 0$, be the coefficients of the expansion of G_q over the Hermite polynomials $\{H_r, r \geq 0\}$ (which are defined in such a way that $\mathbb{E}[H_k(X)H_l(X)] = k!$ if $k = l$ and 0 otherwise). Note that since G is a centered even function, $g_r(q) = 0$ for $r = 0, 1$. Then, by Mehler's formula (see e.g. Arcones (1994)), we have

$$L^{-2} 2^{2n\tau_H(q)} \text{var}_M(S_{L,n}(X, q)) = \sum_{r=2}^{\infty} \frac{g_r(q)^2}{r!} \Gamma_n(r, q),$$

with

$$\begin{aligned} \Gamma_n(r, q) &= L^{-2} 2^{2n\tau_H(q)} \text{var}_M \left(\sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} a_{j,k,n,H}^q H_r(Y_{j,k,n}) \right) \\ &= L^{-2} 2^{2n\tau_H(q)} r! \sum_{j_1, j_2=0}^{L-1} \sum_{k, k'=0}^{2^n-1} \rho_{H,n}^r(j_1, j_2, k, k') a_{j_1,k,n,H}^q a_{j_2,k',n,H}^q. \end{aligned}$$

for $r \in \mathbb{N}, r \geq 2$, and the conditional correlations (which are zero if $H = 1/2$) are

$$\rho_{H,n}(j_1, j_2, k, k') = \text{cov}_M(Y_{j_1, k, n}, Y_{j_2, k', n}) = \frac{\mathbb{E}_M[\Delta X_{j_1, k, n} \Delta X_{j_2, k', n}]}{a_{j_1, k, n, H} a_{j_2, k', n, H}}.$$

By Lemma 3.1 in [Ludeña \(2008\)](#), for $j_1 < j_2$ and $k < k'$, we have the bound

$$\rho_{H,n}(j_1, j_2, k, k') \leq \min(1, C|(j_2 - j_1)2^n + (k' - k)|^{2H-2}) \quad (51)$$

for some deterministic constant C .

We start by proving that for $H < 3/4$ and $q < q_\chi$, there exists a constant $\Gamma(r, q)$ such that

$$\lim_{n \rightarrow \infty} 2^{n(2\psi(q) - \psi(2q) + 1 + \chi)} \mathbb{E}[\Gamma_n(r, q)] = \Gamma(r, q). \quad (52)$$

By the scaling property,

$$\mathbb{E}[a_{j, k, n, H}^q] = 2^{-n\zeta_H(q)} m_H(q),$$

with $\zeta_H(q) = qH - \psi(q)$. Thus, denoting $v_\chi(q) = 2\psi(q) - \psi(2q) + 1 + \chi$, by stationarity, we have

$$\begin{aligned} 2^{nv_\chi(q)} \mathbb{E}[\Gamma_n(r, q)] &= r! m_H(2q) + 2^{-n} 2^{n\zeta_H(2q)} r! \sum_{k \neq k'} \mathbb{E}[\rho_{H,n}^r(k, k') a_{0, k, n, H}^q a_{0, k', n, H}^q] \\ &\quad + 2^{-n(1+\chi)} 2^{n\zeta_H(2q)} r! \sum_{j_1 \neq j_2} \sum_{k, k'} \mathbb{E}[\rho_{H,n}^r(j, j', k, k') a_{j, k, n, H}^q a_{j', k', n, H}^q] \quad (53) \end{aligned}$$

Consider the middle term. Recall that

$$\begin{aligned} \rho_{n, H}^r(k, k') a_{0, k, n, H}^q a_{0, k', n, H}^q &= \left\{ \int_{k2^{-n}}^{(k+1)2^{-n}} \int_{k'2^{-n}}^{(k'+1)2^{-n}} |u - v|^{2H-2} M(du) M(dv) \right\}^r \\ &\quad \times \left\{ \int_{k2^{-n}}^{(k+1)2^{-n}} \int_{k2^{-n}}^{(k+1)2^{-n}} |u - v|^{2H-2} M(du) M(dv) \right\}^{(q-r)/2} \\ &\quad \times \left\{ \int_{k'2^{-n}}^{(k'+1)2^{-n}} \int_{k'2^{-n}}^{(k'+1)2^{-n}} |u - v|^{2H-2} M(du) M(dv) \right\}^{(q-r)/2} \end{aligned}$$

Assume that $k < k'$ and denote $\ell = k' - k + 1$. By the scaling property and the stationarity

of the increments of M , we have

$$\begin{aligned}
& \rho_{n,H}^r(k, k') a_{0,k,n,H}^q a_{0,k',n,H}^q \\
& \stackrel{(law)}{=} (\ell 2^{-n})^{r(2H-2)+2r} e^{2r\Omega_{\ell 2^{-n}}} \left\{ \int_0^{1/\ell} \int_{1-1/\ell}^1 |u-v|^{2H-2} M(du) M(dv) \right\}^r \\
& \times (\ell 2^{-n})^{(q-r)(H-1)+q-r} e^{(q-r)\Omega_{\ell 2^{-n}}} \left\{ \int_0^{1/\ell} \int_0^{1/\ell} |u-v|^{2H-2} M(du) M(dv) \right\}^{(q-r)/2} \\
& \times (\ell 2^{-n})^{(q-r)(H-1)+q-r} e^{(q-r)\Omega_{\ell 2^{-n}}} \left\{ \int_{1-1/\ell}^1 \int_{1-1/\ell}^1 |u-v|^{2H-2} M(du) M(dv) \right\}^{(q-r)/2} \\
& = (\ell 2^{-n})^{2qH} e^{2q\Omega_{\ell 2^{-n}}} Q_\ell^r a_\ell^q b_\ell^q
\end{aligned}$$

with

$$\begin{aligned}
a_\ell^2 &= \int_0^{1/\ell} \int_0^{1/\ell} |u-v|^{2H-2} M(du) M(dv) , \\
b_\ell^2 &= \int_{1-1/\ell}^1 \int_{1-1/\ell}^1 |u-v|^{2H-2} M(du) M(dv) , \\
Q_\ell &= \frac{\int_0^{1/\ell} \int_{1-1/\ell}^1 |u-v|^{2H-2} M(du) M(dv)}{a_\ell b_\ell} \leq C \ell^{2H-2}
\end{aligned} \tag{54}$$

for some deterministic constant C . Thus the middle term in (53) can be expressed as

$$\begin{aligned}
& 2r! 2^{n\zeta_H(2q)} \sum_{\ell=1}^{2^{n-1}} (1 - \ell 2^{-n}) (\ell 2^{-n})^{2qH} \mathbb{E}[e^{2q\Omega_{\ell 2^{-n}}}] \mathbb{E}[Q_\ell^r a_\ell^q b_\ell^q] \\
& = 2r! 2^{n\zeta_H(2q)} 2^{-n\{2qH-\psi(2q)\}} \sum_{\ell=1}^{2^{n-1}} (1 - \ell 2^{-n}) \ell^{2qH-\psi(2q)} \mathbb{E}[Q_\ell^r a_\ell^q b_\ell^q] \\
& = 2r! \sum_{\ell=1}^{2^{n-1}} (1 - \ell 2^{-n}) \ell^{\zeta_H(2q)} \mathbb{E}[Q_\ell^r a_\ell^q b_\ell^q]
\end{aligned}$$

This sum has a limit if the series $\sum_{\ell=1}^{\infty} \ell^{\zeta_H(2q)} \mathbb{E}[Q_\ell^r a_\ell^q b_\ell^q]$ is summable. This holds true since applying the bound (54) and Hölder's inequality yields

$$\mathbb{E}[Q_\ell^r a_\ell^q b_\ell^q] \leq C \ell^{r(2H-2)} \mathbb{E}^{1/2}[a_\ell^{2q}] \mathbb{E}^{1/2}[b_\ell^{2q}] .$$

Applying the stationarity of increments and the scaling property of the MRM M yields $\mathbb{E}[a_\ell^{2q}] = \mathbb{E}[b_\ell^{2q}] = \ell^{-\zeta_H(2q)} m_H(q)$, hence

$$\ell^{\zeta_H(2q)} \mathbb{E}[Q_\ell^r a_\ell^q b_\ell^q] \leq C \ell^{r(2H-2)} .$$

Since $r \geq 2$ and $H < 3/4$, the series $\ell^{r(2H-2)}$ is summable, and thus

$$\lim_{n \rightarrow \infty} \sum_{\ell=1}^{2^{n-1}} (1 - \ell 2^{-n}) \ell^{\zeta_H(2q)} \mathbb{E}[Q_\ell^r a_\ell^q b_\ell^q] = \sum_{\ell=1}^{\infty} \ell^{\zeta_H(2q)} \mathbb{E}[Q_\ell^r a_\ell^q b_\ell^q] .$$

Consider now the last term in (53), say RR_n . Using the bound (51), the scaling property, the fact that the $a_{j,k,n,H}$ are 2-dependent, and $H < 3/4$, we have

$$RR_n \leq C 2^{n\{\zeta_H(2q)-2\zeta_H(q)\}} \sum_{j=1}^L \sum_{k=1}^{2^n} (j2^n + k)^{2H-2} = O(2^{n\{2\psi(q)-\psi(2q)\}}) = o(1) . \quad (55)$$

This proves (52).

We now prove that if $H < 3/4$, for each $r \geq 2$,

$$\Gamma_n(r, q) / \mathbb{E}[\Gamma_n(r, q)] \rightarrow 1 , a.s. \quad (56)$$

or equivalently

$$2^{n\{1+\chi-\psi(2q)+2\psi(q)\}} \Gamma_n(r, q) \rightarrow \Gamma(r, q) , a.s. \quad (57)$$

Write $2^{n\{1+\chi-\psi(2q)+2\psi(q)\}} \Gamma_n(r, q) = r!(S_{n,1} + S_{n,1} + S_{n,1})$ with

$$\begin{aligned} S_{n,1} &= 2^{n\tau_H(2q)} L^{-1} \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} a_{j,k,n,H}^{2q} , \\ S_{n,2} &= 2^{n\tau_H(2q)} L^{-1} \sum_{j=0}^{L-1} \sum_{0 \leq k \neq k' < 2^n} \rho_{H,n}^r(j, j, k, k') a_{j,k,n,H}^q a_{j,k',n,H}^q , \\ S_{n,3} &= 2^{n\tau_H(2q)} L^{-1} \sum_{0 \leq j \neq j' < K} \sum_{k,k'=0}^{2^n-1} \rho_{H,n}^r(j', j', k, k') a_{j,k,n,H}^q a_{j',k',n,H}^q . \end{aligned}$$

The bound (55) and a Borel-Cantelli argument implies that $S_{n,3} \rightarrow 0$ almost surely. The bound (49) implies that

$$2^{n\tau_H(2q)} L^{-1} \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} (a_{j,k,n,H}^{2q} - \tilde{a}_{j,k,n,H}^{2q}) \rightarrow 0 \quad a. s. \quad (58)$$

Thus we consider $\tilde{S}_{n,1} = 2^{n\tau_H(2q)} L^{-1} \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} \tilde{a}_{j,k,n,H}^{2q}$. Using Lemmas 5.2 and 5.4 and mimicking the proof of Proposition 3.1, we obtain that $\tilde{S}_{n,1} \rightarrow m_H(2q)$ a. s.

By stationarity and 2-dependence in j , to deal with $S_{n,2}$, as in the proof of Proposition 3.1, it is enough to prove that

$$\mathbb{E} \left[\left| 2^{n\tau_H(2q)} \sum_{0 \leq k \neq k' < 2^n} \rho_{H,n}^r(0, 0, k, k') a_{0,k,n,H}^q a_{0,k',n,H}^q \right|^p \right] = O(2^{(\epsilon\chi-\eta)n}) \quad (59)$$

for some $\eta > 0$. Since all quantities involved are nonnegative, we can use the bound (51), and suffices to obtain a bound for

$$\mathbb{E} \left[\left| 2^{n\tau_H(2q)} \sum_{0 \leq k \neq k' < 2^n} |k - k'|^{r(2H-2)} a_{0,k,n,H}^q a_{0,k',n,H}^q \right|^p \right].$$

Define

$$\tilde{\delta}_k^2 = \int_{\Delta_{n,k}} \int_{\Delta_{n,k}} |u - v|^{2H-2} M_n(du) M_n(dv).$$

Then $\tilde{a}_{0,k,n,H} = \tilde{\delta}_k e^{qw_{l_n}(k2^{-n})}$ and using the bound (50), we obtain

$$\mathbb{E} \left[\left| 2^{n\tau_H(2q)} \sum_{0 \leq k \neq k' < 2^n} |k - k'|^{r(2H-2)} \{a_{0,k,n,H}^q a_{0,k',n,H}^q - \tilde{a}_{0,k,n,H}^q \tilde{a}_{0,k',n,H}^q\} \right|^p \right] = O(2^{-\eta n}).$$

Thus we need to obtain a bound for $\mathbb{E}[S_{n,4}^p]$ where

$$S_{n,4} = 2^{n\tau_H(2q)} \sum_{0 \leq k \neq k' < 2^n} |k - k'|^{r(2H-2)} \tilde{a}_{0,k,n,H}^q \tilde{a}_{0,k',n,H}^q,$$

wich we further decompose as $S_{n,4} = S_{n,5} + S_{n,6}$ with

$$\begin{aligned} S_{n,5} &= 2^{n\tau_H(2q)} \sum_{0 \leq k \neq k' < 2^n} |k - k'|^{r(2H-2)} \{\tilde{\delta}_k^q \tilde{\delta}_{k'}^q - \mathbb{E}[\tilde{\delta}_k^q \tilde{\delta}_{k'}^q]\} e^{qw_{l_n}(k2^{-n}) + qw_{l_n}(k'2^{-n})} \\ S_{n,6} &= 2^{n\tau_H(2q)} \sum_{0 \leq k \neq k' < 2^n} |k - k'|^{r(2H-2)} \mathbb{E}[\tilde{\delta}_k^q \tilde{\delta}_{k'}^q] e^{qw_{l_n}(k2^{-n}) + qw_{l_n}(k'2^{-n})}. \end{aligned}$$

The bound (48) implies that $2^{n\zeta_H(2q)} \mathbb{E}[\tilde{\delta}_k^q \tilde{\delta}_{k'}^q] \mathbb{E}[e^{qw_{l_n}(k2^{-n}) + qw_{l_n}(k'2^{-n})}]$ is uniformly bounded, thus

$$\mathbb{E}[S_{n,6}^p] \leq C \mathbb{E} \left[\left(2^{-n} \sum_{0 \leq k \neq k' < 2^n} |k - k'|^{r(2H-2)} \frac{e^{qw_{l_n}(k2^{-n}) + qw_{l_n}(k'2^{-n})}}{\mathbb{E}[e^{qw_{l_n}(k2^{-n}) + qw_{l_n}(k'2^{-n})}]} \right)^p \right].$$

Applying Lemma 5.2, for p such that $2pq < q_\chi$ and $\epsilon' < p - 1$, we can prove that

$$\mathbb{E} \left[\left(2^{-n} \sum_{0 \leq k \neq k' < 2^n} |k - k'|^{r(2H-2)} \frac{e^{qw_{l_n}(k2^{-n}) + qw_{l_n}(k'2^{-n})}}{\mathbb{E}[e^{qw_{l_n}(k2^{-n}) + qw_{l_n}(k'2^{-n})}]} \right)^p \right] \leq C l_n^{-\{\psi(2pq) - p\psi(2q) - \epsilon'\}}. \quad (60)$$

If $2pq < q_\chi$, we have

$$(1 - \alpha)\{\psi(2pq) - p\psi(2q) - \epsilon'\} - \epsilon\chi \leq (1 - \alpha)\epsilon(1 + \chi) - \epsilon' \leq \epsilon - \epsilon' - \alpha\epsilon(1 + \chi)$$

which can be made negative by choosing ϵ' close enough to ϵ .

To deal with the last term, as in the proof of Proposition 3.1 we use the conditional $2^{\alpha n}$ dependence of the random variables δ_k . We obtain the bound

$$\mathbb{E}[S_{n,5}^p] \leq C 2^{n\{\psi(2pq) - p\psi(2q) - \epsilon\}} = O(2^{n(\epsilon\chi - \eta)})$$

for small some $\eta > 0$. We have proved (59), and thus (56) holds.

We can now define

$$\Gamma(q) = \sum_{r=2}^{\infty} \frac{g_r(q)^2}{r!} \Gamma(r, q) .$$

As $\sum_{r=2}^{\infty} (r!)^{-1} g_r(q)^2 < \infty$ and $\Gamma_n(r, q) \leq \Gamma_n(2, q)$, then by bounded convergence, the previous series is convergent and

$$2^{n(2\psi(q) - \psi(2q) + 1 + \chi)} \text{var}_M \left(L^{-1} 2^{n\tau_H(q)} S_{L,n}(X, q) \right) \rightarrow \Gamma(q) , \quad \text{a.s.}$$

Hence, for $2q < \bar{q}_\chi$, and $1/2 < H < 3/4$, by Chebyshev's inequality and an application of the Borel Cantelli Lemma, we obtain (19). \square

Proof of Lemma 4.3. For $k \geq 1$, denote

$$\begin{aligned} U_{0,k} &= \int_0^{1/k} \int_0^{1/k} |u - v|^{2H-2} M(du) M(dv) , \\ U_{1,k}^* &= \int_0^{1/2k} \int_0^{1/2k} |u - v|^{2H-2} M(du) M(dv) , \\ U_{2,k} &= \int_{1/2k}^{1/k} \int_{1/2k}^{1/k} |u - v|^{2H-2} M(du) M(dv) , \\ V_{0,k} &= \int_{1-1/k}^1 |u - v|^{2H-2} M(du) M(dv) , \\ V_{1,k}^* &= \int_{1-1/k}^{1-1/2k} \int_{1-1/k}^{1-1/2k} |u - v|^{2H-2} M(du) M(dv) , \\ V_{2,k} &= \int_{1-1/2k}^1 \int_{1-1/2k}^1 |u - v|^{2H-2} M(du) M(dv) . \end{aligned}$$

Then, by the scaling property, we have

$$\begin{aligned} 2^{n\zeta_H(2q)} \mathbb{E}[U_{n,0} U_{n,k}] &= k^{\zeta_H(2q)} \text{cov}(U_{0,k}, V_{0,k}) - 2^{\tau_H(q)} (k - 1/2)^{\zeta_H(2q)} \text{cov}(U_{0,k}, V_{1,k}) \\ &\quad - 2^{\tau_H(q)} k^{\zeta_H(2q)} \{ \text{cov}(U_{0,k}, V_{2,k}) - \text{cov}(U_{1,k}, V_{0,k}) + \text{cov}(U_{2,k}, V_{0,k}) \} \\ &\quad + 2^{2\tau_H(q)} (k - 1/2)^{\zeta_H(2q)} \{ \text{cov}(U_{1,k}, V_{1,k}) + \text{cov}(U_{2,k}, V_{1,k}) \} \\ &\quad + 2^{2\tau_H(q)} k^{\zeta_H(2q)} \{ \text{cov}(U_{1,k}, V_{2,k}) + \text{cov}(U_{2,k}, V_{2,k}) \} . \end{aligned}$$

All the covariance terms are of the same order, and as in the proof of Lemma 3.3, we will only consider the first one. Denote $l = 1 - 2/k$ and define the measure M_l and all other quantities as in the proof of Lemma 3.3. By the same type of decompositions as in the proof of Lemma 3.3, defining

$$\begin{aligned}\zeta_{k,H} &= \int_0^{1/k} \int_0^{1/k} |u - v|^{2H-2} M_l(du) M_l(dv) , \\ \xi_{k,H} &= \int_{1-1/k}^1 \int_{1-1/k}^1 |u - v|^{2H-2} M_l(du) M_l(dv) , \\ \tilde{a}_k(q) &= \mathbb{E}[\zeta_{K,h}^q] , \quad \tilde{M}_l(du) = \tilde{a}_k(q)^{-1/2} M_l(du) , \\ \tilde{\alpha}_k &= \int_0^{1/k} \int_0^{1/k} |u - v|^{2H-2} \{\pi_k(u) + \pi_k(v)\} M_l(du) M_l(dv) , \\ \tilde{b}_k(q) &= \mathbb{E}[\xi_{K,h}^q] , \quad \tilde{M}'_l(du) = \tilde{b}_k(q)^{-1/2} M_l(du) , \\ \tilde{\beta}_k &= \int_{1-1/k}^1 \int_{1-1/k}^1 |u - v|^{2H-2} \{\pi'_k(u) + \pi'_k(v)\} M'_l(du) M'_l(dv) ,\end{aligned}$$

it can be shown that

$$\begin{aligned}\mathbb{E}[U_{0,k}^q] &= e^{2\psi(q)} \tilde{a}_k(q) \{1 + O(k^{-1})\} + q e^{2\psi(q)} \mathbb{E}[\zeta_{k,H}^q \tilde{\alpha}_k] , \\ \mathbb{E}[V_{0,k}^q] &= e^{2\psi(q)} \tilde{b}_k(q) \{1 + O(k^{-1})\} + q e^{2\psi(q)} \tilde{b}_k(q) \mathbb{E}[\xi_{k,H}^q \tilde{\beta}_k] , \\ \mathbb{E}[U_{0,k}^q V_{0,k}^q] &= e^{4\psi(q)} \tilde{a}_k(q) \tilde{b}_k(q) \{1 + O(k^{-1})\} + q e^{2\psi(q)} \{ \tilde{b}_k(q) \mathbb{E}[\zeta_{k,H}^q \tilde{\alpha}_k] + \tilde{a}_k(q) \mathbb{E}[\xi_{k,H}^q \tilde{\beta}_k] \} .\end{aligned}$$

This proves (21). \square

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